Kangweon-Kyungki Math. Jour. **11** (2003), No. 2, pp. 85–91

SOME PROPERTIES OF WEIGHTED HARMONIC BERGMAN FUNCTIONS ON HALF-SPACES

Young-Chae Nah and HeungSu Yi

Abstract. On the setting of the upper half-space of the euclidean space $\mathbf{R}^n$, we show some properties of weighted harmonic Bergman functions.

1. Introduction

For a fixed positive integer $n > 1$, let $\mathbf{H} = \mathbf{R}^{n-1} \times \mathbf{R}_+$ be the upper half-space where $\mathbf{R}_+$ denotes the set of all positive real numbers. We write point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \le p < \infty$, let $b^p_\alpha(\mathbf{H})$ denote *weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on $\mathbf{H}$ such that

$$||u||_{L^p_\alpha} := \left(\int_{\mathbf{H}} |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty,$$

where $dV_\alpha(z) = z_n^\alpha \, dz$ and dz is the Lebesgue measure on $\mathbf{R}^n$. We let $b^p_\alpha = b^p_\alpha(\mathbf{H})$. Then we can check easily that the space b^p_α is a Banach space with the usual weighted L^p-norm.

In this paper, we show some properties of b^p_α as stated below. In section 2, we review some basic results of the extended Poisson kernel. In section 3 we show the b^1_α-cancellation property, i.e., If $u \in b^1_\alpha$, then $\int_{\mathbf{H}} u(z) \, dV_\alpha(z) = 0$. we also find a necessary and sufficient condition for

Received April 24, 2003.
2000 Mathematics Subject Classification: Primary 31B05, Secondary 31B10, 30D55.
Key words and phrases: Weighted Harmonic Bergman Function, Upper Half-Space.
The first author was partially supported by Mokwon University during sabbatical year in 2002 and the second author was partially supported by the Research Grant of Kwangwoon Univeristy in 2002 and KOSEF 2000-1-10100-001-3.

the space b^p_α to have a positive harmonic function and then we show that b^p_α is not contained in b^q_α if q is different from p.

Constants. Throughout the paper we use the same letter C to denote various constants which may change at each occurrence. The constant C may often depend on the dimension n and some other parameters, but it is always independent of particular points or parameters under consideration. For nonnegative quantities A and B, we often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some *inessential* positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

2. Preliminary Results

Let $P(z, w)$ be the extended Poisson kernel on $\mathbf{H}$, i.e.,

$$P_z(w) := P(z, w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z - \overline{w}|^n} \tag{2.1}$$

where $V(B)$ is the volume of the unit ball in $\mathbf{R}^n$, $z \in \mathbf{H}$, $w \in \overline{\mathbf{H}} = \mathbf{H} \cup \partial\mathbf{H}$, and $\overline{w} = (w', -w_n)$. Here $\partial\mathbf{H} = \mathbf{R}^{n-1}$ denote the boundary of $\mathbf{H}$. Note that for each fixed $w \in \overline{\mathbf{H}}$, $P(z, w)$ is a positive and harmonic function on $\mathbf{H}$ as a function of z. Note also that for each $z \in \mathbf{H}$ and for every $w \in \overline{\mathbf{H}}$,

$$\int_{\partial\mathbf{H}} P(z, w)\, dw' = 1. \tag{2.2}$$

Also, we can show from (**??**) that for nonnegative integer k,

$$D^k_n P(z, 0) = \frac{f_k(z)}{|z|^{n+2k}}, \tag{2.3}$$

where f_k is a homogeneous polynomial of degree $1 + k$.

The poisson integral of $f \in L^p(\partial\mathbf{H})$, for $1 \le p \le \infty$, is the function $P[f]$ on $\mathbf{H}$ defined by

$$P[f](z) = \int_{\partial\mathbf{H}} P(z, t) f(t)\, dt.$$

Let k be a nonnegative integer. If $u \in b^p_\alpha$, then we know from the mean value property, Jensen's inequality and then Cauchy's estimate that

$$|D^k_n u(z)| \lesssim z_n^{-(n+\alpha)/p-k}$$

for each $z \in \mathbf{H}$. This shows that if $u \in b^p_\alpha$, then u is a bounded harmonic function on every proper half-space contained in $\mathbf{H}$. Thus we have

$$P[u(\cdot, z_n)](z', \delta) = u(z', z_n + \delta) \tag{2.4}$$

for $\delta > 0$. (See [?] and [?] for details and related facts.)

3. Some Properties of b^p_α

In this section, we show some properties of the weighted harmonic Bergman functions. First we show the b^1_α-cancellation property. To do so, we need a lemma.

LEMMA 1. *If $u \in b^1_\alpha$, then $\widetilde{u}$ is decreasing on $(0, \infty)$, where*

$$\widetilde{u}(\delta) = \int_{\partial\mathbf{H}} \big|u(t, \delta)\big|\, dt$$

for $\delta > 0$.

Proof. Suppose $0 < \delta_1 < \delta_2$. Then we know from (**??**) that

$$u(t, \delta_2) = P[u(\cdot, \delta_1)](t, \delta_2 - \delta_1).$$

Therefore we have

$$\big|u(t, \delta_2)\big| \le \int_{\partial\mathbf{H}} \big|u(s, \delta_1)\big| P\big((t, \delta_2 - \delta_1), s\big)\, ds.$$

Note that

$$P\big((t, \delta_2 - \delta_1), s\big) = P\big((s, \delta_2 - \delta_1), t\big)$$

for every $s, t \in \partial\mathbf{H}$. Integrating with respect to t and then using Fubini's theorem, we see from (**??**) that

$$\begin{aligned}
\widetilde{u}(\delta_2) &= \int_{\partial H} \big|u(t, \delta_2)\big|\, dt \\
&\le \int_{\partial H} \big|u(s, \delta_1)\big| \int_{\partial H} P\big((s, \delta_2 - \delta_1), t\big)\, dt\, ds \\
&= \widetilde{u}(\delta_1),
\end{aligned}$$

as desired. Therefore the proof is complete. □

As the above proof shows, the result of Lemma **??** holds if we only assume that u equals the Poisson integral of its boundary values on every proper half-space contained in $\mathbf{H}$.

Now we are ready to prove the following theorem.

THEOREM 2. *If $u \in b^1_\alpha$, then*

$$\int_{\partial\mathbf{H}} u(t,\delta)\,dt = 0$$

for each $\delta > 0$.

Proof. Fix $\delta > 0$. Then we know from Lemma **??** that $u(\cdot,\delta) \in L^1(\partial\mathbf{H})$. Also, we know from (**??**) that

$$u(z', z_n + \delta) = P[u(\cdot,\delta)](z)$$

for every $z \in \mathbf{H}$. Therefore we have from Fubini's theorem and (**??**) that

$$\begin{aligned}
\int_{\mathbf{H}} u(z', z_n + \delta)\,dz &= \int_0^\infty \int_{\partial\mathbf{H}} \int_{\partial\mathbf{H}} P(z,t)u(t,\delta)\,dt\,dz'\,dz_n \\
&= \int_0^\infty \int_{\partial\mathbf{H}} \left(\int_{\partial\mathbf{H}} P\big((t,z_n),z'\big)\,dz' \right) u(t,\delta)\,dt\,dz_n \\
&= \int_0^\infty \int_{\partial\mathbf{H}} u(t,\delta)\,dt\,dz_n. \qquad (3.1)
\end{aligned}$$

Because the inner integral in (**??**) is independent of z_n, we must have

$$\int_{\partial\mathbf{H}} u(t,\delta)\,dt = 0.$$

Therefore the proof is complete. □

As a corollary to the above theorem, we easily get the following b^1_α-cancellation property.

COROLLARY 3. *If $u \in b^1_\alpha$, then*

$$\int_{\mathbf{H}} u(z)\,dV_\alpha(z) = 0.$$

Now we find a necessary and sufficient condition for the space b^p_α to have a positive harmonic function. This means that certain b^p_α spaces do not contain any positive functions on the upper half-space and we can not have this property on bounded domains.

THEOREM 4. *b^p_α contains a positive harmonic function if and only if $p > (n+\alpha)/(n-1)$.*

Proof. Suppose that $p > (n+\alpha)/(n-1)$. Let $z_0 = (0,1)$ and let $u(z) = P(z, z_0)$ for $z \in \mathbf{H}$. Then clearly, u is a positive harmonic function on $\mathbf{H}$. Note from (**??**) that

$$\left|P(z,z_0)\right|^{p-1} \lesssim (z_n+1)^{-(n-1)(p-1)}.$$

Therefore we have from (**??**) that

$$\begin{aligned} \|u\|^p_{L^p_\alpha} &= \int_{\mathbf{H}} |u(z)|^p z_n^\alpha \, dz \\ &\lesssim \int_0^\infty \int_{\partial\mathbf{H}} P(z,z_0)\, dz' \frac{z_n^\alpha}{(z_n+1)^{(n-1)(p-1)}}\, dz_n \\ &= \int_0^\infty \frac{z_n^\alpha}{(z_n+1)^{(n-1)(p-1)}}\, dz_n. \qquad (3.2) \end{aligned}$$

Because $\alpha > -1$ and $(n-1)(p-1) - \alpha > 1$, the integral in (**??**) is finite. Hence we see that $u \in b^p_\alpha$ as desired.

Conversely, suppose $u \in b^p_\alpha$ is positive on $\mathbf{H}$. Then we know from [**?**] that

$$u(z) = cz_n + P[\mu](z)$$

for all $z \in \mathbf{H}$, where c is a nonnegative constant and μ is a positive Borel measure on $\partial\mathbf{H}$ satisfying

$$\int_{\partial\mathbf{H}} \frac{d\mu(t)}{(1+|t|)^n} < \infty.$$

Because $u \in b^p_\alpha$, we must have $c = 0$. Since u is a positive harmonic function, μ can not be the zero measure and so we can choose a compact set $K \subset \partial\mathbf{H}$ satisfying $\mu(K) > 0$. Let $R = \max\{\, |t| : t \in K\}$. Then we see that

$$\begin{aligned} u(z) &\geq \frac{2z_n}{nV(B)\big(|z|+R\big)^n}\mu(K) \\ &\gtrsim \frac{z_n}{(|z|+1)^n} \end{aligned}$$

on $\mathbf{H}$. Thus we have from (**??**) that

$$\begin{aligned}\infty &> \int_{\mathbf{H}} |u(z)|^p z_n^\alpha \, dz \\ &\gtrsim \int_{\mathbf{H}} \frac{z_n^{p+\alpha}}{(|z|+1)^{np}} \, dz \\ &\gtrsim \int_0^\infty \frac{z_n^{p+\alpha}}{(z_n+1)^{np-n+1}} \, dz.\end{aligned}$$

Because $u \in b^p$, we must have $np - n + 1 - p - \alpha > 1$, i.e., $p > n/(n-1)$. This completes the proof. □

We close this paper by showing that on $\mathbf{H}$, no Bergman space is properly contained in another.

THEOREM 5. *If $p \neq q$, then b^p_α does not contain b^q_α.*

Proof. Suppose to the contrary that $b^p_\alpha \subset b^q_\alpha$. Because convergence in any Bergman space implies uniform convergence on compact subsets, the closed graph theorem shows that the identity map from b^p_α to b^q_α is bounded. Thus there exists a positive constant C satisfying

$$\|v\|_{L^q_\alpha} \le C \|v\|_{L^p_\alpha} \tag{3.3}$$

for all $v \in b^p_\alpha$.

To show that (**??**) fails, we choose a nonnegative integer k large enough so that

$$(n+k-1)p > n + \alpha, \qquad (n+k-1)q > n+\alpha. \tag{3.4}$$

Set $u(z) = D_n^k P(z, 0)$ for $z \in \mathbf{H}$. Then clearly u is harmonic on $\mathbf{H}$ and we see from (**??**) that

$$u(z) = \frac{f_k(z)}{|z|^{n+2k}}$$

for some homogenous polynomial of degree $k+1$. Let $u_\delta(z) = u\big(z + (0,\delta)\big)$ for $\delta > 0$. Then clearly u_δ is harmonic on $\mathbf{H}$. We also see from the homogeneity of f that

$$\begin{aligned}\|u_\delta\|^p_{L^p_\alpha} &= \int_{\mathbf{H}} \frac{\big|f\big(z+(0,\delta)\big)\big|^p}{\big|z+(0,\delta)\big|^{(n+2k)p}} z_n^\alpha \, dz \\ &= \frac{\delta^{n+(k+1)p+\alpha}}{\delta^{(n+2k)p}} \int_{\mathbf{H}} \frac{\big|f\big(z+(0,1)\big)\big|^p}{\big|z+(0,1)\big|^{(n+2k)p}} z_n^\alpha \, dz. \qquad (3.5)\end{aligned}$$

We see from (**??**) that the integral in (**??**) is finite. Thus,

$$\|u_\delta\|_{L^p_\alpha} \approx \delta^{(n+\alpha)/p-n-k+1}$$

and similarly we have

$$\|u_\delta\|_{L^q_\alpha} \approx \delta^{(n+\alpha)/q-n-k+1}.$$

Therefore

$$\frac{\|u_\delta u\|_{L^q_\alpha}}{\|u_\delta u\|_{L^p_\alpha}} \approx \delta^{(n+\alpha)(1/q-1/p)} \tag{3.6}$$

for all $\delta > 0$. Because $p \neq q$, the right side of (**??**) is not a bounded function of δ. Thus (**??**) fails and the proof is complete. □

References

[1] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Springer-Verlag, New York, 1992.

[2] E. Stein and G. Weiss, *Fourier analysis on euclidean spaces* , Princeton University Press, Princeton, 1971.

Department of Mathematics
Mokwon University
Taejon, 302-729, Korea
E-mail: ycnah@mwus.mokwon.ac.kr

Department of Mathematics
Kwangwoon University
Seoul 139–701, Korea
E-mail: hsyi@kwangwoon.ac.kr