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ON A LIMIT CLASS OF LORENTZ OPERATOR IDEALS

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Abstract. We give an extensive presentation of results about the behaviour of the approximation operator ideals $\mathcal{L}_{\infty,q}$ in connection with the Lorentz operator ideals $\mathcal{L}_{p,q}$.

1. Introduction

Schatten-von Neumann operator ideals

$$S_{p,q} = \{T \in \mathcal{L}(H) : (\sum_{n=1}^{\infty}[n^{1/p-1/q}s_n(T)]^q)^{1/q} < \infty\},$$

$0 < p < \infty$, $0 < q \leq \infty$, where $s_n(T)$ is the n-th singular number of the operator T acting on a Hilbert space H, have played an important role as the historical starting point of the theory of operators.

In order to treat certain problems of perturbation theory and invariant subspaces, V. Macaev [6] introduced the operator ideals $S_{\infty,1} = \{T \in \mathcal{L}(H) : \sum_{n=1}^{\infty} n^{-1}s_n(T) < \infty\}$. Later on, the operator ideals $S_{\infty,q} = \{T \in \mathcal{L}(H) : (\sum_{n=1}^{\infty} n^{-1}s_n(T)^q)^{1/q} < \infty\}$, $0 < q < \infty$, appeared naturally in the work of V. Peller [7] on the averaging projection onto the set of Hankel matrices.

It is well-known that approximation numbers coincide with singular numbers for operators acting between Hilbert spaces. So we can extend the operator ideals described above to the class of all Banach spaces by setting for $0 < p < \infty$ and $0 < q \leq \infty$,$\mathcal{L}_{p,q} = \{T \in \mathcal{L} : (\sum_{n=1}^{\infty}[n^{1/p-1/q}a_n(T)]^q)^{1/q} < \infty\}$, and for $0 < q < \infty$,$\mathcal{L}_{\infty,q} = \{T \in$

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$\mathcal{L} : (\sum_{n=1}^{\infty} n^{-1} a_n(T)^q)^{1/q} < \infty\}$. Here $a_n(T)$ is the n-th approximation number of the operator T acting between Banach spaces.

Based on the theory of Lorentz operator ideals $\mathcal{L}_{p,q}$, many striking results on eigenvalue distributions of abstract operators on Banach spaces were established and successfully applied to various types of integral operators. Some important properties of Lorentz operator ideals $\mathcal{L}_{p,q}$ follow from the fact that they can be regarded as approximation spaces.

F. Cobos and I. Resina [4] established some results about the operator ideals $\mathcal{L}_{\infty,q}$. It turned out that the general theory of Lorentz operator ideals $\mathcal{L}_{p,q}$ does not cover the limiting case of operator ideals $\mathcal{L}_{\infty,q}$ by taking $p = \infty$.

In this paper we survey geometric structures of the operator ideals $\mathcal{L}_{\infty,q}$.

We first give a representation theorem for operators belonging to the operator ideals $\mathcal{L}_{\infty,q}$ in terms of finite rank operators. By virtue of this result we obtain a multiplication formula.

Next we deal with the stability under tensor products of the operator ideals $\mathcal{L}_{\infty,q}$. And then we describe the behaviour under interpolation of the operator ideals $\mathcal{L}_{\infty,q}$.

Finally we investigate the relationship between the operator ideals $\mathcal{L}_{\infty,q}$ and entropy operator ideals $\mathcal{L}_{\infty,q}^{(e)}$ generated by entropy numbers.

2. Definitions and Notation

We present some of the definitions and notation to be used. Throughout this paper E and F denote Banach spaces.

If $x = (x_i)$ is a bounded sequence then we put $s_n(x) = \inf\{\sigma \geq 0 : \text{card}(i : |x_i| \geq \sigma) < n\}$. $\big(s_n(x)\big)$ is called the non-increasing rearrangement of x. Let $0 < p < \infty$ and $0 < q \leq \infty$. Then the Lorentz sequence space $\ell_{p,q}$ consists of all sequences $x = (x_i)$ having a finite quasi-norm

$$\|x|\ell_{p,q}\| = \begin{cases} \big(\sum_{n=1}^{\infty} [n^{1/p-1/q}\, s_n(x)]^q\big)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_n [n^{1/p} s_n(x)] & \text{if } q = \infty. \end{cases}$$

Notation. (1) $\mathcal{L}(E, F)$ denotes the set of all bounded linear operators from E into F.

(2) $\mathcal{F}(E,F)$ denotes the set of all finite rank operators from E into F.

(3) $\mathcal{K}(E,F)$ denotes the set of all compact operators from E into F.

(4) The closed unit ball of E is denoted by B_E.

The n-th approximation number of $T \in \mathcal{L}(E,F)$ is defined by

$$a_n(T) = \inf\{\|T - L\| : L \in \mathcal{F}(E,F),\ \operatorname{rank}(L) < n\}.$$

An operator $T \in \mathcal{L}(E,F)$ is said to be of approximation type $\ell_{p,q}$ if $(a_n(T)) \in \ell_{p,q}$. The Lorentz operator ideal $\mathcal{L}_{p,q}(E,F)$ consists of these operators. For $T \in \mathcal{L}_{p,q}(E,F)$, we define $\|T|\mathcal{L}_{p,q}\| = \|(a_n(T))|\ell_{p,q}\|$.

For $0 < q < \infty$, the approximation operator ideal $\mathcal{L}_{\infty,q}(E,F)$ is the set of all operators $T \in \mathcal{L}(E,F)$ which have a finite quasi-norm $\|T|\mathcal{L}_{\infty,q}\| = (\sum_{n=1}^{\infty} n^{-1} a_n(T)^q)^{1/q}$.

Let (α_n) and (β_n) be non-negative real-valued sequences. Then $\alpha_n \prec \beta_n$ means that $\alpha_n \le C\,\beta_n$ for $n = 1, 2, \cdots$, where the constant $C > 0$ may depend on various parameters but not on the index n. We write $\alpha_n \sim \beta_n$ if $\alpha_n \prec \beta_n$ and $\beta_n \prec \alpha_n$.

By a cross norm τ we mean a norm which is simultaneously defined on all algebraic tensor products $E \otimes F$ such that $\tau(x \otimes y) = \|x\|_E\,\|y\|_F$ for $x \in E$ and $y \in F$. We denote the space $E \otimes F$ equipped with τ by $E \otimes_\tau F$ and its completion by $E \hat{\otimes}_\tau F$.

The algebraic tensor product of the operators $S \in \mathcal{L}(E,F)$ and $T \in \mathcal{L}(E_0,F_0)$ is the linear operator $S \otimes T$ from $E \otimes E_0$ into $F \otimes F_0$ defined uniquely by $S \otimes T(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n Sx_i \otimes Ty_i,\ x_1, \cdots, x_n \in E,\ y_1, \cdots, y_n \in E_0$. A cross-norm is called a tensor norm provided that for all such maps the following holds : $\tau(\sum_{i=1}^n Sx_i \otimes Ty_i) \le \|S\|\|T\|\,\tau(\sum_{i=1}^n x_i \otimes y_i)$. In this case $S \otimes T$ admits a unique τ-continuous extension acting from $E \hat{\otimes}_\tau E_0$ into $F \hat{\otimes}_\tau F_0$ which is denoted by $S \hat{\otimes}_\tau T$.

An operator ideal $\mathcal{U}$ is said to be stable with respect to a tensor norm τ if $S \in \mathcal{U}(E,F)$ and $T \in \mathcal{U}(E_0,F_0)$ imply $S \hat{\otimes}_\tau T \in \mathcal{U}(E \hat{\otimes}_\tau E_0, F \hat{\otimes}_\tau F_0)$.

Let (E_0, E_1) be a couple of quasi-Banach spaces. We consider the functional $K(t,x,E_0,E_1) = K(t,x) = \inf\{\|x_0\|_{E_0} + t\,\|x_1\|_{E_1} : x_0 \in E_0, x_1 \in E_1, x = x_0 + x_1\}$ on $E_0 + E_1$. If $0 < \theta < 1$ and $0 < q \le \infty$ then the real interpolation space $(E_0, E_1)_{\theta,q}$ consists of all elements

$x \in E_0 + E_1$ which have a finite quasi-norm

$$\|x\|_{(E_0,E_1)_{\theta,q}} = \|x\|_{\theta,q} = \begin{cases} \left(\int_0^\infty [t^{-\theta}K(t,x)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } \ 0<q<\infty, \\ \sup_t [t^{-\theta}K(t,x)] & \text{if } \ q=\infty. \end{cases}$$

Let $0 < q < \infty$. We denote by $\mathcal{L}^{\infty}_{\infty,q}(E,F)$ the set of all operators $T \in \mathcal{L}(E,F)$ which have a finite quasi-norm $\|T \,|\, \mathcal{L}^{\infty}_{\infty,q}\| = \inf\big(\sup_{n\geq 0}[2^{n/q}\|T_n\|\,]\big)$, where the infimum is taken over all representations $T = \sum_{n=0}^{\infty} T_n$ with $\operatorname{rank} T_n \leq 2^{(2^n)}$ and $\sup_{n\geq 0}[2^{n/q}\|T_n\|] < \infty$.

For every operator $T \in \mathcal{L}(E,F)$ the n-th outer entropy number $e_n(T)$ is defined to be the infimum of all $\sigma \geq 0$ such that there are elements $y_1, \cdots, y_q \in F$ with $q \leq 2^{n-1}$ and $T(B_E) \subseteq \cup_{i=1}^{q}\{y_i + \sigma B_F\}$.

For $0 < q < \infty$, the space $\mathcal{L}^{(e)}_{\infty,q}(E,F)$ is defined to consist of all operators $T \in \mathcal{L}(E,F)$ which have a finite quasi-norm $\|T \,|\, \mathcal{L}^{(e)}_{\infty,q}\| = \big(\sum_{n=1}^{\infty} n^{-1}e_n(T)^q\big)^{1/q}$.

An operator ideal $\mathcal{U}$ is surjective if for every surjection $Q \in \mathcal{L}(E_0,E)$ and every operator $T \in \mathcal{L}(E,F)$ it follows from $TQ \in \mathcal{U}(E_0,F)$ that $T \in \mathcal{U}(E,F)$.

The surjective hull $\mathcal{U}^s$ of an operator ideal $\mathcal{U}$ is the smallest surjective operator ideal containing $\mathcal{U}$. In case of a quasi-Banach operator ideal $\mathcal{U}$ the surjective hull $\mathcal{U}^s$ of $\mathcal{U}$ becomes a quasi-Banach operator ideal if it is endowed with the quasi-norm

$$\|T : E \to F \,|\, \mathcal{U}^s\| = \inf\{\|S|\mathcal{U}\| : T(B_E) \subseteq S(B_G), \text{where } S \in \mathcal{U}(G,F)\}.$$

Let $\mathcal{U}$ be any quasi-Banach operator ideal. The n-th generalized entropy number of $T \in \mathcal{U}^S(E,F)$ is defined by

$$e_n(T|\mathcal{U}) = \inf\{\|S|\mathcal{U}\| : T(B_E) \subseteq \cup_{i=1}^{q}[y_i + S(B_G)], \\ \text{where } y_1,\cdots,y_q \in F \text{ with } q \leq 2^{n-1} \text{ and } S \in \mathcal{U}(G,F)\}.$$

Let $\mathcal{U}$ be an operator ideal. A sequence (y_n) in F is called $\mathcal{U}$-convergent to zero if there is an operator $S \in \mathcal{U}(G,F)$ with the following property : Given $\epsilon > 0$ there exists a natural number n_ϵ such that $y_n \in \epsilon \cdot S(B_G)$ for all $n > n_\epsilon$.

Let $\mathcal{U}$ be an operator ideal. A subset M of F is called $\mathcal{U}$-compact if $M \subseteq \{y \in F : y = \sum_{i=1}^{\infty} \lambda_i y_i, \sum_{i=1}^{\infty} |\lambda_i| \leq 1\}$, where (y_i) is a sequence in F which is $\mathcal{U}$-convergent to zero.

Let $\mathcal{U}$ be an operator ideal. An operator $T \in \mathcal{U}^S(E,F)$ is called $\mathcal{U}$-compact if for each bounded subset U of E, $T(U)$ is $\mathcal{U}$-compact in F. The set of these operators is denoted by $\mathcal{U}_{c_0}^{(e)}$.

A quasi-Banach operator ideal $\mathcal{U}$ is approximative if $\mathcal{F}(E,F)$ is dense in every component $\mathcal{U}(E,F)$.

Let $\mathcal{U}$ be any quasi-Banach operator ideal. The n-th generalized approximation number of $T \in \mathcal{U}(E,F)$ is defined by

$$a_n(T\,|\,\mathcal{U}) = \inf\{\|T - L\,|\,\mathcal{U}\| : L \in \mathcal{F}(E,F),\ \operatorname{rank}(L) < n\}.$$

An operator $T \in \mathcal{U}(E,F)$ with $\lim_{n\to\infty} a_n(T|\mathcal{U}) = 0$ is said to be $\mathcal{U}$-approximable. The collection of all $\mathcal{U}$-approximable operators is denoted by $\mathcal{U}_{c_0}^{(a)}$.

Given a family of Banach spaces E_n with $n \in \mathbb{N}$, the direct sum $[\ell_2, E_n]$ consists of all sequences $(x_n)_{n=1}^{\infty}$ such that $x_n \in E_n, n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \|x_n\|_{E_n}^2 < \infty$.

3. Results

We begin by showing that the lexicographical order of the scale of Lorentz operator ideals $\mathcal{L}_{p,q}$ can be transferred to the limiting case $p = \infty$.

Proposition 1. $\mathcal{L}_{\infty,p} \subset \mathcal{L}_{\infty,q}$ *for* $0 < p < q < \infty$.

Proof. We select $T \in \mathcal{L}_{\infty,p}$. An appeal to the monotonicity of approximation numbers reveals that

$$\begin{aligned}
\|T|\mathcal{L}_{\infty,q}\| &\leq [\sum_{n=1}^{\infty} (a_n(T)(\log n)^{1/p})^{q-p} a_n(T)^p n^{-1}]^{1/q} \\
&\leq [\sup_{n\geq 1} \{a_n(T)(\sum_{k=1}^{n} k^{-1})^{1/p}\}]^{(q-p)/q} \, [\sum_{n=1}^{\infty} a_n(T)^p n^{-1}]^{1/q} \\
&\leq (\sum_{k=1}^{\infty} a_k(T)^p k^{-1})^{1/p\cdot(q-p)/q} \|T|\mathcal{L}_{\infty,p}\|^{p/q} = \|T|\mathcal{L}_{\infty,p}\|.
\end{aligned}$$

This gives us the desired inclusion. □

F. Cobos and I. Resina [4] showed that the representation theorem for Lorentz operator ideals $\mathcal{L}_{p,q}$ fails to be true in the limiting case $p = \infty$. The following lemma enables us to establish the representation theorem for the operator ideals $\mathcal{L}_{\infty,q}$.
From now on, we write ν_n instead of $2^{(2^n)}, n = 0, 1, 2, \cdots$.

Lemma 1. *Let $0 < q < \infty$. If $T \in \mathcal{L}(E, F)$ then $\sum_{n=1}^{\infty} a_n(T)^q n^{-1} \sim a_1(T)^q + \sum_{n=0}^{\infty} 2^n a_{\nu_n}(T)^q$.*

Proof. We deduce from the monotonicity of approximation numbers that

$$\sum_{n=1}^{\infty} a_n(T)^q n^{-1} = a_1(T)^q + \sum_{k=0}^{\infty} \sum_{n=\nu_k}^{\nu_{k+1}-1} n^{-1} a_n(T)^q$$

$$\leq a_1(T)^q + \sum_{k=0}^{\infty} (\sum_{n=\nu_k}^{\nu_{k+1}-1} n^{-1}) a_{\nu_k}(T)^q \leq C(a_1(T)^q + \sum_{k=0}^{\infty} 2^k a_{\nu_k}(T)^q).$$

On the one hand, we have

$$a_1(T)^q + \sum_{n=0}^{\infty} 2^n a_{\nu_n}(T)^q \leq C_0(a_1(T)^q + a_2(T)^q + \sum_{k=0}^{\infty} 2^k a_{\nu_{k+1}}(T)^q)$$

$$\leq C_1(a_1(T)^q + a_2(T)^q + \sum_{k=0}^{\infty} (\sum_{n=\nu_k+1}^{\nu_{k+1}} n^{-1}) a_{\nu_{k+1}}(T)^q)$$

$$\leq C_2 \sum_{n=1}^{\infty} a_n(T)^q n^{-1}.$$

This completes the proof. □

Theorem 1. *Let $0 < q < \infty$. An operator $T \in \mathcal{L}(E, F)$ belongs to $\mathcal{L}_{\infty,q}(E, F)$ if and only if there exists a sequence $(T_n)_{n=0}^{\infty} \subset \mathcal{F}(E, F)$ with $\mathrm{rank}(T_n) \leq 2^{(2^n)}$ such that $T = \sum_{n=0}^{\infty} T_n$ converges in the operator norm and $\sum_{n=0}^{\infty} 2^n \|T_n\|^q < \infty$. Moreover, $\|T|\mathcal{L}_{\infty,q}\|^{\mathrm{rep}} = \inf\{[\sum_{n=0}^{\infty} 2^n \|T_n\|^q]^{1/q}\}$, where the infimum is taken over all possible representations, defines an equivalent quasi-norm on $\mathcal{L}_{\infty,q}$.*

Proof. If $T \in \mathcal{L}_{\infty,q}(E,F)$ then we choose $L_n \in \mathcal{F}(E,F)$ such that $\|T - L_n\| \leq 2a_{\nu_n}(T)$ and $\operatorname{rank}(L_n) < 2^{(2^n)} = \nu_n$ for $n = 0, 1, 2, \cdots$. Define $T_0 = 0, T_1 = L_0$ and $T_n = L_{n-1} - L_{n-2}$ for $n = 2, 3, \cdots$. Then we have $\operatorname{rank}(T_n) \leq \operatorname{rank}(L_{n-1}) + \operatorname{rank}(L_{n-2}) < \nu_{n-1} + \nu_{n-2} < \nu_n$, $\|T_1\| \leq \|T - L_0\| + \|T\| \leq 2a_2(T) + a_1(T) \leq 3a_1(T)$, and $\|T_n\| \leq \|L_{n-1} - T\| + \|T - L_{n-2}\| \leq 4a_{\nu_{n-2}}(T)$, $n = 2, 3, \cdots$. Since $(a_n(T))$ converges to zero, it follows that $T = \lim_k L_k = \sum_{k=0}^{\infty} T_k$.

We invoke lemma 1 to infer that

$$\sum_{n=0}^{\infty} 2^n \|T_n\|^q \leq 2 \cdot 3^q a_1(T)^q + \sum_{n=0}^{\infty} 2^{(n+2)} 4^q a_{\nu_n}(T)^q$$
$$\leq C_0 (a_1(T)^q + \sum_{n=0}^{\infty} 2^n a_{\nu_n}(T)^q) \leq C_1 \sum_{n=1}^{\infty} a_n(T)^q n^{-1}.$$

Additionally, it turns out that $\|T|\mathcal{L}_{\infty,q}\|^{\mathrm{rep}} \leq C_1^{1/q} \|T|\mathcal{L}_{\infty,q}\|$.

We now verify the sufficiency of the given condition. To this end, assume that $T \in \mathcal{L}(E,F)$ admits a representation with the properties stated above. Then we have $\operatorname{rank}(\sum_{k=0}^{n-1} T_k) \leq \sum_{k=0}^{n-1} \nu_k < \nu_n$, $n = 1, 2, \cdots$. Therefore $a_{\nu_n}(T) \leq \|T - \sum_{k=0}^{n-1} T_k\| \leq \sum_{k=n}^{\infty} \|T_k\|$, $n = 1, 2, \cdots$, and $a_{\nu_0}(T) \leq a_1(T) = \|T\| \leq \sum_{k=0}^{\infty} \|T_k\|$.

Fix p and λ such that $0 < p < \min(1, q)$ and $0 < \lambda < 1/q$. Define s by $1/p = 1/s + 1/q$. We apply Hölder's inequality to get that

$$a_{\nu_n}(T) \leq \sum_{k=n}^{\infty} \|T_k\| \leq (\sum_{k=n}^{\infty} \|T_k\|^p)^{1/p}$$
$$\leq (\sum_{k=n}^{\infty} 2^{-\lambda k s})^{1/s} (\sum_{k=n}^{\infty} 2^{\lambda k q} \|T_k\|^q)^{1/q} \leq K_o 2^{-\lambda n} (\sum_{k=n}^{\infty} 2^{\lambda k q} \|T_k\|^q)^{1/q}.$$

Combining this estimate with lemma 1 we obtain

$$
\begin{aligned}
[\sum_{n=1}^{\infty} a_n(T)^q n^{-1}]^{1/q} &\le K_1[a_1(T)^q + \sum_{n=0}^{\infty} 2^n a_{\nu_n}(T)^q]^{1/q} \\
&\le K_1[a_1(T)^q + K_0^q \sum_{n=0}^{\infty} 2^{n(1-\lambda q)}(\sum_{k=n}^{\infty} 2^{\lambda k q}\|T_k\|^q)]^{1/q} \\
&= K_1[a_1(T)^q + K_0^q \sum_{k=0}^{\infty}(\sum_{n=0}^{k} 2^{n(1-\lambda q)})2^{\lambda k q}\|T_k\|^q]^{1/q} \\
&\le K_2[a_1(T)^q + \sum_{k=0}^{\infty} 2^k\|T_k\|^q]^{1/q}.
\end{aligned}
$$

This proves that $T \in \mathcal{L}_{\infty,q}(E,F)$ and $\|T|\mathcal{L}_{\infty,q}\| \le K\|T|\mathcal{L}_{\infty,q}\|^{\text{rep}}$. □

In the next theorem we see that the multiplication formula for Lorentz operator ideals $\mathcal{L}_{p,q}$ remains true for the limiting case $p = \infty$.

THEOREM 2. *If* $0 < q_0, q_1 < \infty$ *and* $1/q = 1/q_0 + 1/q_1$ *then* $\mathcal{L}_{\infty,q_0} \circ \mathcal{L}_{\infty,q_1} = \mathcal{L}_{\infty,q}$.

Proof. Let $T \in \mathcal{L}_{\infty,q_1}(E,F)$ and $S \in \mathcal{L}_{\infty,q_0}(F,G)$. Applying Hölder's inequality and the multiplicativity of approximation numbers, we derive

$$
\begin{aligned}
\|ST|\mathcal{L}_{\infty,q}\| = (\sum_{n=1}^{\infty} a_n(ST)^q n^{-1})^{1/q} &\le C(\sum_{n=1}^{\infty} a_{2n-1}(ST)^q n^{-1})^{1/q} \\
&\le C(\sum_{n=1}^{\infty}[n^{-1/q_0}a_n(s)n^{-1/q_1}a_n(T)]^q)^{1/q} \\
&\le C(\sum_{n=1}^{\infty} a_n(S)^{q_0} n^{-1})^{1/q_0}(\sum_{n=1}^{\infty} a_n(T)^{q_1} n^{-1})^{1/q_1}.
\end{aligned}
$$

Hence $ST \in \mathcal{L}_{\infty,q}(E,G)$. This yields that $\mathcal{L}_{\infty,q_0} \circ \mathcal{L}_{\infty,q_1} \subseteq \mathcal{L}_{\infty,q}$.

To verify the reverse inclusion, take $R \in \mathcal{L}_{\infty,q}(E,G)$. We consider a representation $R = \sum_{k=0}^{\infty} R_k$ such that $R_k \in \mathcal{F}(E,G)$, $\text{rank}(L_k) \le 2^{(2^k)} = \nu_k$, and $\sum_{k=0}^{\infty} 2^k\|R_k\|^q < \infty$. Choose factorizations $R_k =$

S_kT_k with $T_k \in \mathcal{L}(E, F_k)$, $S_k \in \mathcal{L}(F_k, G)$, $\|T_k\| = \|R_k\|^{q/q_1}$, $\|S_k\| = \|R_k\|^{q/q_0}$ and $\dim(F_k) \leq \nu_k$. We take F to be the ℓ_2 direct sum $[\ell_2, F_k]$ of countably many copies of F_k. Let $J_k \in \mathcal{L}(F_k, F)$ and $Q_k \in \mathcal{L}(F, F_k)$ denote the canonical injections and surjections, respectively. Notice that $\operatorname{rank}(S_k\, Q_k) \leq \nu_k$, $\sum_{k=0}^{\infty} 2^k \|S_k\, Q_k\|^{q_0} \leq \sum_{k=0}^{\infty} 2^k \|R_k\|^q$, and $\operatorname{rank}(J_k\, T_k) \leq \nu_k$, $\sum_{k=0}^{\infty} 2^k \|J_k\, T_k\|^{q_1} \leq \sum_{k=0}^{\infty} 2^k \|R_k\|^q$. We take account of theorem 1 to conclude that $S = \sum_{k=0}^{\infty} S_k Q_k \in \mathcal{L}_{\infty,q_0}(F, G)$ and $T = \sum_{k=0}^{\infty} J_k T_k \in \mathcal{L}_{\infty,q_1}(E, F)$. As a result $R = ST \in \mathcal{L}_{\infty,q_0} \circ \mathcal{L}_{\infty,q_1}(E, G)$. This implies that $\mathcal{L}_{\infty,q} \subseteq \mathcal{L}_{\infty,q_0} \circ \mathcal{L}_{\infty,q_1}$. □

A. Pietsch [10] and H. König [5] showed that Lorentz operator ideals $\mathcal{L}_{p,q}$ fail to be tensor-stable. However, the theorem given below shows that the operator ideals $\mathcal{L}_{\infty,q}$ are stable with respect to tensor norms.

THEOREM 3. *Let $0 < q < \infty$. If $S \in \mathcal{L}_{\infty,q}(E, F)$ and $T \in \mathcal{L}_{\infty,q}(E_0, F_0)$ then $S \hat{\otimes}_\tau T \in \mathcal{L}_{\infty,q}(E \hat{\otimes}_\tau E_0, F \hat{\otimes}_\tau F_0)$ for any tensor norm τ.*

Proof. Given any $\epsilon > 0$, we pick $S_0 \in \mathcal{F}(E, F)$ and $T_0 \in \mathcal{F}(E_0, F_0)$ such that $\operatorname{rank}(S_0) < 2^n$ and $\|S - S_0\| < a_{2^n}(S) + \epsilon$, $\operatorname{rank}(T_0) < 2^n$ and $\|T - T_0\| < a_{2^n}(T) + \epsilon$. Since $\operatorname{rank}(S_0 \hat{\otimes}_\tau T_0) = \operatorname{rank}(S_0) \cdot \operatorname{rank}(T_0) < 2^{2n}$, we get

$$\begin{aligned}
a_{2^{2n}}(S \hat{\otimes}_\tau T) &\leq \|S \hat{\otimes}_\tau T - S_0 \hat{\otimes}_\tau T_0\| = \|(S - S_0) \hat{\otimes}_\tau T + S_0 \hat{\otimes}_\tau (T - T_0)\| \\
&\leq \|S - S_0\| \|T\| + \|S_0\| \|T - T_0\| \\
&\leq (a_{2^n}(S) + \epsilon)\|T\| + (a_{2^n}(S) + \epsilon + \|S\|)(a_{2^n}(T) + \epsilon) \\
&\leq (a_{2^n}(S) + \epsilon)\|T\| + (2\|S\| + \epsilon)(a_{2^n}(T) + \epsilon).
\end{aligned}$$

Passing with ϵ to zero we obtain $a_{2^{2n}}(S \hat{\otimes}_\tau T) \leq 2[a_{2^n}(S)\|T\| + \|S\| a_{2^n}(T)]$.

This enables us to obtain the following estimate

$$\|S\hat{\otimes}_\tau T|\mathcal{L}_{\infty,q}\| = [\sum_{n=1}^{\infty} a_n(S\hat{\otimes}_\tau T)^q n^{-1}]^{1/q} \le [\sum_{n=0}^{\infty} a_{2^n}(S\hat{\otimes}_\tau T)^q]^{1/q}$$

$$\le 2^{\frac{1}{q}}[\sum_{n=0}^{\infty} a_{2^{2n}}(S\hat{\otimes}_\tau T)^q]^{\frac{1}{q}} \le 2^{\frac{1}{q}} \cdot 2\,[\sum_{n=0}^{\infty}(a_{2^n}(S)\|T\| + \|S\|a_{2^n}(T))^q]^{\frac{1}{q}}$$

$$\le 2^{\frac{1}{q}+1}\max\{1, 2^{\frac{1}{q}-1}\}\,[(\sum_{n=0}^{\infty} a_{2^n}(S)^q)^{\frac{1}{q}}\|T\| + \|S\|(\sum_{n=0}^{\infty} a_{2^n}(T)^q)^{\frac{1}{q}}]$$

$$\le C(\sum_{n=0}^{\infty} a_n(S)^q n^{-1})^{\frac{1}{q}} \cdot (\sum_{n=0}^{\infty} a_n(T)^q n^{-1})^{\frac{1}{q}} = C\|S|\mathcal{L}_{\infty,q}\| \cdot \|T|\mathcal{L}_{\infty,q}\|.$$

Thus we have $S\hat{\otimes}_\tau T \in \mathcal{L}_{\infty,q}(E\hat{\otimes}_\tau E_0, F\hat{\otimes}_\tau F_0)$. □

Now we describe how the scale of operator ideals $\mathcal{L}_{\infty,q}$ behaves under interpolation.

THEOREM 4. *Let* $0 < q_0, q_1 < \infty$ *and* $0 < \theta < 1$. *If* $1/q = (1-\theta)/q_0 + \theta/q_1$ *then* $(\mathcal{L}_{\infty,q_0}\mathcal{L}_{\infty,q_1})_{\theta,q} = \mathcal{L}_{\infty,q}$. *The quasi-norms on both sides are equivalent.*

Proof. We divide the proof into two steps.

Step 1. The first step is to verify the following formula : $\mathcal{L}_{\infty,q} = (\mathcal{L}, \mathcal{L}^{\infty}_{\infty,p})_{\theta,q}$ for $0 < p < \infty,\ 0 < \theta < 1$ and $1/q = \theta/p$.

Let us take $T \in \mathcal{L}_{\infty,q}(E,F)$. For $n = 0, 1, 2, \cdots$, we can find $L_n \in \mathcal{F}(E,F)$ such that $\|T - L_n\| \le 2a_{\nu_n}(T)$ and $\text{rank}(L_n) < \nu_n$. Define $T_0 = 0, T_1 = L_0$ and $T_n = L_{n-1} - L_{n-2}$ for $n = 2, 3 \cdots$. Then we have $T = \sum_{k=0}^{\infty} T_k$ with $\|T_1\| \le 3a_1(T)$, $\|T_n\| \le 4a_{\nu_{n-2}}(T)$, $n = 2, 3 \cdots$, and $\|T - \sum_{k=0}^{n} T_k\| \le 2a_{\nu_{n-1}}(T)$, $n = 1, 2 \cdots$. Fix ρ and λ such that $0 < \rho < \min(1, q)$ and $1/q < \lambda < 1/p$. Define s by $1/\rho = 1/s + 1/q$.

We make use of Hölder's inequality to derive that

$$K(2^{-n/p},T) \leq \|T-\sum_{k=0}^{n+1}T_k\| + 2^{-n/p}\|\sum_{k=0}^{n+1}T_k|\mathcal{L}_{\infty,p}^{\infty}\|$$

$$\leq 2a_{\nu_n}(T) + 2^{-n/p}(\sum_{k=0}^{n+1}2^{k\rho/p}\|T_k\|^{\rho})^{1/\rho}$$

$$\leq C_0[a_{\nu_n}(T) + 2^{-n/p}(a_1(T)^{\rho} + \sum_{k=0}^{n}2^{k(1/p-\lambda)\rho}2^{k\lambda\rho}a_{\nu_k}(T)^{\rho})^{1/\rho}]$$

$$\leq C_0[a_{\nu_n}(T) + 2^{-n/p}(1+\sum_{k=0}^{n}2^{k(\frac{1}{p}-\lambda)s})^{\frac{1}{s}}(a_1(T)^q + \sum_{k=0}^{n}2^{k\lambda q}a_{\nu_k}(T)^q)^{\frac{1}{q}}]$$

$$\leq C_1[a_{\nu_n}(T) + 2^{-n\lambda}(a_1(T)^q + \sum_{k=0}^{n}2^{k\lambda q}a_{\nu_k}(T)^q)^{\frac{1}{q}}].$$

Using this inequality, together with lemma 3.1.3 of [1] and lemma 1, we get

$$\|T\|_{\theta,q} = (\int_0^{\infty}[\tau^{-\theta}K(\tau,t)]^q\frac{d\tau}{\tau})^{1/q}$$

$$\leq C_2(\sum_{n=0}^{\infty}[2^{n\theta/p}K(2^{-n/p},T)]^q)^{1/q} = C_2(\sum_{n=0}^{\infty}2^nK(2^{-n/p},T)^q)^{1/q}$$

$$\leq C_3(\sum_{n=0}^{\infty}2^na_{\nu_n}(T)^q + \sum_{n=0}^{\infty}2^{n(1-\lambda q)}[a_1(T)^q + \sum_{k=0}^{n}2^{k\lambda q}a_{\nu_k}(T)^q])^{1/q}$$

$$\leq C_4(\sum_{n=0}^{\infty}2^na_{\nu_n}(T)^q + a_1(T)^q + \sum_{k=0}^{\infty}2^{k\lambda q}a_{\nu_k}(T)^q\sum_{n=k}^{\infty}2^{n(1-\lambda q)})^{1/q}$$

$$\leq C_5(a_1(T)^q + \sum_{n=0}^{\infty}2^na_{\nu_n}(T)^q)^{1/q} \leq C_6(\sum_{n=1}^{\infty}a_n(T)^qn^{-1})^{1/q}.$$

Hence $T \in (\mathcal{L}(E,F),\mathcal{L}_{\infty,p}^{\infty}(E,F))_{\theta,q}$. This proves that

$$\mathcal{L}_{\infty,q} \subseteq (\mathcal{L},\mathcal{L}_{\infty,p}^{\infty})_{\theta,q}$$

.

To show that we have equality, take $T \in (\mathcal{L}(E,F), \mathcal{L}^{\infty}_{\infty,p}(E,F))_{\theta,q}$. Let $T = T_0 + T_1$, where $T_0 \in \mathcal{L}(E,F)$ and $T_1 \in \mathcal{L}^{\infty}_{\infty,p}(E,F)$. We consider a representation $T_1 = \sum_{n=0}^{\infty} L_n$ such that $\mathrm{rank}(L_n) \leq \nu_n$ and $\sup_{n\geq 0}\{2^{n/p}\|L_n\|\} \leq 2\|T_1|\mathcal{L}^{\infty}_{\infty,p}\|$. Since $\mathrm{rank}(\sum_{k=0}^{n-1} L_k) \leq \sum_{k=0}^{n-1} \nu_k < \nu_n$, for any $\rho > 0$ small enough, we have

$$a_{\nu_n}(T_1) \leq \|T_1 - \sum_{k=0}^{n-1} L_k\| = \|\sum_{k=n}^{\infty} L_k\| \leq (\sum_{k=n}^{\infty} 2^{-k\rho/p}2^{k\rho/p}\|L_k\|^{\rho})^{1/\rho}$$
$$\leq C_0 2^{-n/p} \sup_{k\geq 0}\{2^{k/p}\|L_k\|\} \leq 2C_0 2^{-n/p}\|T_1|\mathcal{L}^{\infty}_{\infty,p}\|.$$

Therefore from the additivity of approximation numbers it follows that

$$a_{\nu_n}(T) \leq a_1(T_0) + a_{\nu_n}(T_1) \leq C_1(\|T_0\| + 2^{-n/p}\|T_1|\mathcal{L}^{\infty}_{\infty,p}\|)$$

and thus $a_{\nu_n}(T) \leq C_1 K(2^{-n/p}, T)$.
While, for any $\rho > 0$ small enough, we obtain

$$a_1(T_1) = \|T_1\| \leq (\sum_{k=0}^{\infty} 2^{-k\rho/p}2^{k\rho/p}\|L_k\|^{\rho})^{1/\rho}$$
$$\leq C_2 \sup_{k\geq 0}\{2^{k/p}\|L_k\|\} \leq 2C_2\|T_1|\mathcal{L}^{\infty}_{\infty,p}\|.$$

It takes another appeal to the additivity of approximation numbers to yield that

$$a_1(T) \leq a_1(T_0) + a_1(T_1) = \|T_0\| + a_1(T_1) \leq C_3(\|T_0\| + \|T_1|\mathcal{L}^{\infty}_{\infty,p}\|)$$

and so $a_1(T) \leq C_3\, K(1,T)$.
Applying lemma 3.1.3 of [1] and lemma 1 again, we have

$$(\sum_{n=1}^{\infty} a_n(T)^q n^{-1})^{1/q} \leq C_4(a_1(T)^q + \sum_{n=0}^{\infty} 2^n a_{\nu_n}(T)^q)^{1/q}$$
$$\leq C_5(K(1,T)^q + \sum_{n=0}^{\infty}[2^{n/q}K(2^{-n/p},T)]^q)^{1/q}$$
$$\leq C_6(\sum_{n=0}^{\infty}[2^{n\theta/p}K(2^{-n/p},T)]^q)^{1/q} \leq C_7(\int_0^{\infty} [\tau^{-\theta}K(\tau,T)]^q \frac{d\tau}{\tau})^{1/q}.$$

Hence $T \in \mathcal{L}_{\infty,q}(E,F)$. This proves that $(\mathcal{L}, \mathcal{L}^{\infty}_{\infty,p})_{\theta,q} \subseteq \mathcal{L}_{\infty,q}$.
Step 2. We improve the result of the preceding step by the reiteration property. Choose p such that $0 < p < \min(q_0, q_1)$. Define θ_0 and θ_1 by $1/q_0 = \theta_0/p$ and $1/q_1 = \theta_1/p$, respectively. Step 1 allows us to obtain that $\mathcal{L}_{\infty,q_0} = (\mathcal{L}, \mathcal{L}^{\infty}_{\infty,p})_{\theta_0,q_0}$ and $\mathcal{L}_{\infty,q_1} = (\mathcal{L}, \mathcal{L}^{\infty}_{\infty,p})_{\theta_1,q_1}$. Theorem 3.11.5 of [1] and Step 1 inform us that

$$\begin{aligned}(\mathcal{L}_{\infty,q_0}, \mathcal{L}_{\infty,q_1})_{\theta,q} &= \big((\mathcal{L}, \mathcal{L}^{\infty}_{\infty,p})_{\theta_0,q_0}, (\mathcal{L}, \mathcal{L}^{\infty}_{\infty,p})_{\theta_1,q_1}\big)_{\theta,q} \\ &= (\mathcal{L}, \mathcal{L}^{\infty}_{\infty,p})_{p/q,q} = \mathcal{L}_{\infty,q}\end{aligned}$$

because $p/q = (1-\theta)p/q_0 + \theta p/q_1 = (1-\theta)\theta_0 + \theta\theta_1$. This ends the proof. □

We pass to the discussion of the link between the operator ideals $\mathcal{L}_{\infty,q}$ and $\mathcal{L}^{(e)}_{\infty,q}$.

THEOREM 5. *Let $0 < q < \infty$. Then $\mathcal{L}_{\infty,q} \subseteq \mathcal{L}^{(e)}_{\infty,q}$.*

Proof. We assume that $T \in \mathcal{L}_{\infty,q}(E,F)$. A result due to B.Carl [2] guarantees the existence of a constant C_0 such that

$$\sup_{1\le k\le n} k^{1/q} e_k(T) \le C_0 \sup_{1\le k\le n} k^{1/q} a_k(T), \quad n = 1, 2 \cdots .$$

Thus

$$n\, e_n(T)^q \le C_0^q \sup_{1\le k\le n} k a_k(T)^q \le C_0^q \sum_{j=1}^{n} a_j(T)^q, \;\; n = 1, 2, \cdots .$$

This leads us to have that

$$\begin{aligned}\|T|\mathcal{L}^{(e)}_{\infty,q}\| &= [\sum_{n=1}^{\infty} n^{-1} e_n(T)^q]^{1/q} \le C_0[\sum_{n=1}^{\infty} n^{-2}(\sum_{k=1}^{n} a_k(T)^q)]^{1/q} \\ &= C_0[\sum_{k=1}^{\infty}(\sum_{n=k}^{\infty} n^{-2}) a_k(T)^q]^{1/q} \le C_1[\sum_{k=1}^{\infty} k^{-1} a_k(T)^q]^{1/q}.\end{aligned}$$

Hence $T \in \mathcal{L}^{(e)}_{\infty,q}(E,F)$. This yields the desired inclusion. □

Finally we strengthen the above inclusion. For this purpose we need the following preliminary results. As an immediate consequence of the representation theorem we obtain the next result.

PROPOSITION 2. *If $0 < q < \infty$ then the operator ideal $\mathcal{L}_{\infty,q}$ is approximative.*

Proof. Let us take any $T \in \mathcal{L}_{\infty,q}(E,F)$. We consider any representation $T = \sum_{k=0}^{\infty} T_k$ such that $\text{rank}(T_k) \le \nu_k$ and $\sum_{k=0}^{\infty} 2^k \|T_k\|^q < \infty$. Given $\epsilon > 0$, we choose a natural number n_0 with $(\sum_{k=n_0}^{\infty} 2^k \|T_k\|^q)^{1/q} \le \epsilon$. We call on theorem 1 to obtain that

$$\|T - \sum_{k=0}^{n} T_k | \mathcal{L}_{\infty,q}\| \le C \, \|T - \sum_{k=0}^{n} T_k | \mathcal{L}_{\infty,q}\|^{\text{rep}} \le C\epsilon \quad \text{for } n \ge n_0.$$

This completes the proof. □

We derive the useful characterization of the ideal of $\mathcal{U}$-compact operators.

PROPOSITION 3. *Let $\mathcal{U}$ be a quasi-Banach operator ideal. Then $\mathcal{U}_{c_0}^{(e)} = (\mathcal{U}_{c_0}^{(a)})^s$.*

Proof. Let $T \in \mathcal{U}(E,F)$. Given $\epsilon > 0$, we can find $L \in \mathcal{F}(E,F)$ such that $\text{rank}(L) < m$ and $\|T - L|\mathcal{U}\| < a_m(T|\mathcal{U}) + \epsilon$. It takes an appeal to the additivity of the generalized entropy numbers to see that

$$\begin{aligned} &e_{(m-1)^2+1}(T|\mathcal{U}) \le \kappa \, [\|T - L|\mathcal{U}^s\| + e_{(m-1)^2+1}(L|\mathcal{U})] \\ &\le \kappa \, [\|T - L|\mathcal{U}\| + e_{(m-1)^2+1}(L|\mathcal{U})] < \kappa \, [a_m(T|\mathcal{U}) + e_{(m-1)^2+1}(L|\mathcal{U}) + \epsilon]. \end{aligned}$$

Take the canonical factorization $L = \tilde{L}Q$, where $Q \in \mathcal{L}(E, E/\text{ker}L)$ denotes the canonical surjection and $\tilde{L} \in \mathcal{L}(E/\text{ker}L, F)$ is the operator induced by L. We put $E_0 = E/\text{ker}L$. The multiplicativity of the generalized entropy numbers ensures that $e_{(m-1)^2+1}(L|\mathcal{U}) \le \|\tilde{L}|\mathcal{U}^s\| \cdot e_{(m-1)^2+1}(Q)$. The surjectivity of the ideal quasi-norm $\|\cdot|\mathcal{U}^s\|$ makes that $\|\tilde{L}|\mathcal{U}^s\| = \|\tilde{L}Q|\mathcal{U}^s\| \le \|L|\mathcal{U}\|$. Also the surjectivity of the entropy numbers permits us to have that $e_{(m-1)^2+1}(I_{E_0}) = e_{(m-1)^2+1}(I_{E_0}Q) = e_{(m-1)^2+1}(Q)$. Since $\dim(E_0) < m$, it follows from theorem 12.1.10 and proposition 12.1.13 of [8] that $e_{(m-1)^2+1}(I_{E_0}) \le 4/2^{m-1}$. Whence $e_{(m-1)^2+1}(L|\mathcal{U}) \le \|L|\mathcal{U}\| \cdot 4/2^{m-1}$. Note that

$$\|L|\mathcal{U}\| \le \kappa[\|L - T|\mathcal{U}\| + \|T|\mathcal{U}\|] < \kappa[a_m(T|\mathcal{U}) + \|T|\mathcal{U}\| + \epsilon] \le \kappa[2\|T|\mathcal{U}\| + \epsilon].$$

Combining the preceding inequalities, we obtain

$$e_{(m-1)^2+1}(T|\mathcal{U}) \leq \kappa\,[a_m(T|\mathcal{U}) + 8\kappa/2^{m-1}\|T|\mathcal{U}\| + \epsilon(1 + 4\kappa/2^{m-1})].$$

Let ϵ tend to zero to get

$$(*) \qquad e_{(m-1)^2+1}(T|\mathcal{U}) \leq \kappa\,[a_m(T|\mathcal{U}) + 8\kappa/2^{m-1}\|T|\mathcal{U}\|].$$

Now we assume that $T \in \mathcal{U}_{c_0}^{(a)}(E,F)$. Then it follows from $(*)$ and $\lim_{m\to\infty} a_m(T|\mathcal{U}) = 0$ that $\lim_{m\to\infty} e_{(m-1)^2+1}(T|\mathcal{U}) = 0$ and hence $T \in \mathcal{U}_{c_0}^{(e)}(E,F)$. This assures us that $\mathcal{U}_{c_0}^{(a)} \subseteq \mathcal{U}_{c_0}^{(e)}$. The surjectivity of the operator ideal $\mathcal{U}_{c_0}^{(e)}$ tells us that $(\mathcal{U}_{c_0}^{(a)})^s \subseteq \mathcal{U}_{c_0}^{(e)}$.

To obtain the converse inclusion, we select $T \in \mathcal{U}_{c_0}^{(e)}(E,F)$. Then the $\mathcal{U}$-compactness of $T(B_E)$ alerts us to the fact that $T(B_E) \subseteq \{y \in F : y = \sum_{i=1}^{\infty} \lambda_i y_i, \sum_{i=1}^{\infty} |\lambda_i| \leq 1\} = M$, where (y_i) is a sequence in F which is $\mathcal{U}$-convergent to zero. Furthermore, we have $y_i = Sz_i$, where $S \in \mathcal{U}(G,F)$ and (z_i) is a sequence in G converging to zero. This sequence permits us to define an operator $R : \ell_1 \to G$ via $R(\lambda_i) = \sum_{i=1}^{\infty} \lambda_i z_i$ for all $(\lambda_i) \in \ell_1$. It is obvious that $SR(B_{\ell_1}) = M$. We consider the operator $R_n : \ell_1^n \to G$ which is given by $R_n(\lambda_i) = \sum_{i=1}^{n} \lambda_i z_i$ for all $(\lambda_i)_{i=1}^n \in \ell_1^n$. It follows from $\|S(R - R_n)|\mathcal{U}\| \leq \|S|\mathcal{U}\|\|R - R_n\| \leq \|S|\mathcal{U}\| \sup_{i>n} \|z_i\|$ that $\lim_{n\to\infty} \|SR - SR_n|\mathcal{U}\| = 0$. Consequently $SR \in \mathcal{U}_{c_0}^{(a)}(\ell_1, F)$. As $T(B_E) \subseteq SR(B_{\ell_1}) = M$, we have $T \in (\mathcal{U}_{c_0}^{(a)})^s(E,F)$. This gives that $\mathcal{U}_{c_0}^{(e)} \subseteq (\mathcal{U}_{c_0}^{(a)})^s$. □

Using the specific properties of the ideal of compact operators we draw the factorization formula for the ideal of $\mathcal{U}$-compact operators.

Proposition 4. *Let $\mathcal{U}$ be a quasi-Banach operator ideal. Then $\mathcal{U}_{c_0}^{(e)} = \mathcal{U}^s \circ K$ or $\mathcal{U}_{c_0}^{(e)} = \mathcal{U}_{c_0}^{(e)} \circ K$, respectively.*

Proof. Let $T \in K(E,F)$ and $S \in \mathcal{U}^s(F,G)$. Then $T(B_E)$ is a precompact subset of F and so $S(T(B_E))$ is $\mathcal{U}$-compact. This means that $ST \in \mathcal{U}_{c_0}^{(e)}(E,G)$. Hence we have $\mathcal{U}^s \circ K \subseteq \mathcal{U}_{c_0}^{(e)}$.

We now assume that $R \in \mathcal{U}_{c_0}^{(e)}(E,G)$. Since $R(B_E)$ is $\mathcal{U}$-compact, we invoke the fact that $R(B_E) \subseteq \{z \in G : z = \sum_{i=1}^{\infty} \lambda_i z_i, \sum_{i=1}^{\infty} |\lambda_i| \leq 1\} = N$,where (z_i) is a sequence in G which is $\mathcal{U}$-convergent to zero.

Moreover, we have $z_i = Sy_i$, where $S \in \mathcal{U}(F,G)$ and (y_i) is a sequence in F converging to zero. we construct a sequence (ρ_i) of real numbers such that $\rho_i \geq 1$, $\lim_i \rho_i = \infty$ and $\lim_i \rho_i y_i = 0$. We set $u_i = \rho_i y_i$, $i = 1, 2, \cdots$. Let $M = \{y \in F : y = \sum_{i=1}^{\infty} \lambda_i u_i, \sum_{i=1}^{\infty} |\lambda_i| \leq 1\}$. Then M is a precompact subset of F and thus $S(M)$ is an $\mathcal{U}$-compact subset of G. Define $G_0 = \{z \in G : z = \tau w, \text{where}\, w \in S(M)\}$, where G_0 is equipped with the norm $\|z\|_{G_0} = \inf\{\tau > 0 : z = \tau w, \text{where}\, w \in S(M)\}$. Then G_0 is a Banach space. Notice that $R(B_E) \subseteq N \subseteq S(M)$. This guarantees the factorization of $R = R_1 R_0$ through a Banach space G_0, where $R_0 \in \mathcal{L}(E, G_0)$ is an operator which is given by $R_0 x = Rx$ for all $x \in E$ and $R_1 \in \mathcal{L}(G_0, G)$ is the identity map. Since $z_i = Sy_i = \frac{1}{\rho_i} Su_i$ and $Su_i \in S(M)$, it follows that $\|z_i\|_{G_0} \leq \frac{1}{\rho_i}$ and so $\lim_i \|z_i\|_{G_0} = 0$. Accordingly N is a precompact subset of G_0 and hence $R(B_E)$ is precompact as well. This forces that $R_0 \in \mathcal{K}(E, G_0)$. Observe that $R_1(B_{G_0}) \subset (1+\epsilon)S(M)$ and $S(M)$ is an $\mathcal{U}$-compact subset of G. This indicates that $R_1 \in \mathcal{U}_{c_0}^{(e)}(G_0, G)$. Therefore $R = R_1 R_0 \in \mathcal{U}_{c_0}^{(e)} \circ K(E,G)$. Combining the first inclusion we arrive at $\mathcal{U}^s \circ K \subseteq \mathcal{U}_{c_0}^{(e)} \subseteq \mathcal{U}_{c_0}^{(e)} \circ K$. As a consequence $\mathcal{U}_{c_0}^{(e)} = \mathcal{U}^s \circ K$ or $\mathcal{U}_{c_0}^{(e)} = \mathcal{U}_{c_0}^{(e)} \circ K$, respectively. □

We are now in a position to improve theorem 5.

THEOREM 6. *Let* $0 < q < \infty$. *Then* $\mathcal{L}_{\infty,q} \subseteq \mathcal{L}_{\infty,q}^{(e)} \circ K$.

Proof. We use theorem 5 and the surjectivity of the operator ideal $\mathcal{L}_{\infty,q}^{(e)}$ to see that $(\mathcal{L}_{\infty,q})^s \subseteq \mathcal{L}_{\infty,q}^{(e)}$. Thanks to propositions 2,3 and 4, we obtain $(\mathcal{L}_{\infty,q})^s = ((\mathcal{L}_{\infty,q})_{c_0}^{(a)})^s = (\mathcal{L}_{\infty,q})_{c_0}^{(e)} = (\mathcal{L}_{\infty,q})^s \circ K$. As a result we have $\mathcal{L}_{\infty,q} \subseteq (\mathcal{L}_{\infty,q})^s \subseteq \mathcal{L}_{\infty,q}^{(e)} \circ K$. □

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