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A FIXED POINT THEOREM FOR ASYMPTOTICALLY C-CONTRACTIVE MAPPINGS

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Abstract. Under some asymptotical conditions we obtain a fixed point theorem for a mapping(not necessarily nonexpansive) from an unbounded closed convex subset of a reflexive Banach space into itself. Also we give an example for applying it.

1. Introduction

Luc[4] extended a famous result by Browder[1] and Göhde[2] and Kirk[3] by replacing the boundedness of domains with the condition of an asymptotically compactness. Recently Penot [5] obtained a fixed point theorem for a nonexpansive mapping without the boundedness and the condition of an asymptotically compactness. He introduced the following definition of an asymptotically contractive mapping which bears on the behavior of the map at infinity.

Definition 1. [5] Let C be a subset of a Banach space X. We say that $f : C \to X$is *asymptotically contractive on* C if there exists $x_0 \in C$ such that

$$\limsup_{x \in C, \|x\| \to \infty} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < 1$$

We note that this condition is independent of the choice of x_0. And the followings are equivalent conditions of the definition of asymptotically contractive on C. $f : C \to X$ is asymptotically contractive on C iff there exists $x_0 \in C, c \in (0, 1)$ and $R > 0$ such that for all $x \in C, \|x\| \geq R$

$$\|f(x) - f(x_0)\| \leq c\|x - x_0\|$$

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iff there exists $c \in (0,1)$ and $R > 0$ such that for all $x \in C$, $\|x\| \geq R$

$$\|f(x)\| \leq c\|x\|.$$

Furthermore a fixed point set of an asymptotically contractive map on C is empty or bounded and any linear convex combinations of two asymptotically contractive maps on C is also asymptotically contractive map on C . Recall that a map $f : C \to X$ is nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in C$. It is said to be demi-closed if its graph is sequentially closed in the product of the weak topology on C with the norm topology on X. Penot obtained the following fixed point theorem(Proposition 2 in [5]).

THEOREM 2. [5] *Let Xbe a reflexive Banach space and let C be a closed convex subset of X. Let $f : C \to X$ be a nonexpansive map which is asymptotically contractive on C and such that $f(C) \subset C$ and $I - f$is demi-closed. Then f has a fixed point.*

2. Main Result and Example

Before we we state the main result we present the asymptotically c-contractiveness of a (not necessarily nonexpansive) mapping. Let us note $\mathbb{R}^+$is the set of nonnegative real numbers.

DEFINITION 3. Let C be a subset of a Banach space X. We say that $f : C \to X$ is asymptotically c-contractive on C if there is a function $c : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$such that

(1) c is symmetric (i.e., for all $s, t \in \mathbb{R}^+$, $c(s,t) = c(t,s)$)
(2) for all $s \in \mathbb{R}^+$,
$$\limsup_{t \to \infty} c(s,t) < 1$$
(3) there exists $r_0 \geq 0$ such that for all $s, t \geq r_0$, $c(s,t) \leq 1$
(4) there exists $r_1 > r_0$ such that if $\|x\| \geq r_0$ and $x \in C$, then $\|f(x)\| \geq r_1$
(5) for all $x, y \in C$,
$$\|f(x) - f(y)\| \leq c(\|x\|, \|y\|)\|x - y\|.$$

We state the main fixed point theorem in this paper.

THEOREM 4. *Let X be a reflexive Banach space and let C be a unbounded closed convex subset of X. Let $f : C \to X$ be asymptotically c-contractive on C such that $f(C) \subset C$ and $I - f$ is demi-closed. Then f has a fixed point.*

Proof. Let (t_n)be a sequence in $(0, 1)$ with limit 0 and let $x_0 \in C$. For all $n \geq 1$, let $f_n : C \to X$ be given by

$$f_n(x) := (1 - t_n)f(x) + t_n x_0.$$

Then $f_n : C \to C$ by the convexity of C. And for all sufficiently large n

$$f_n : C - r_0 B_X \longrightarrow C - r_0 B_X,$$

where r_0 is given in the definition of asymptotically c-contractiveness and $B_X = \{x \in C | \|x\| < 1\}$. Indeed for all $x \in C$, $\|x\| \geq r_0$,

$$\begin{aligned} \|f_n(x)\| &= \|(1 - t_n)f(x) + t_n x_0\| \\ &\geq (1 - t_n)\|f(x)\| - t_n\|x_0\| \\ &\geq (1 - t_n)r_1 - t_n\|x_0\| \\ &> r_0 \end{aligned}$$

for all sufficiently large n. Moreover for all $x, y \in C - r_0 B_X$,

$$\begin{aligned} \|f_n(x) - f_n(y)\| &= (1 - t_n)\|f(x) - f(y)\| \\ &\leq (1 - t_n)c(\|x\|, \|y\|)\|x - y\| \\ &\leq (1 - t_n)\|x - y\| \end{aligned}$$

Hence by the Picard-Banach fixed point theorem, f_n has a fixed point $x_n \in C - r_0 B_X$ for all sufficiently large n. We can show that (x_n) is bounded. On the contrary we assume that (x_n) is unbounded. We can choose a subsequence (x_{n_i}) such that $\lim_{n_i \to \infty} \|x_{n_i}\| = \infty$. Then

$$\begin{aligned} \|x_{n_i}\| &= \|(1 - t_{n_i})f(x_{n_i}) + t_{n_i} x_0\| \\ &\leq (1 - t_{n_i})\|f(x_{n_i})\| + t_{n_i}\|x_0\| \end{aligned}$$

and by the condition (2) of asymptotically c-contractiveness there exists $\alpha \in (0, 1)$ such that

$$\|f(x_{n_i}) - f(x_0)\| \leq \alpha\|x_{n_i} - x_0\|$$

for all sufficiently large n_i. Therefore

$$\|x_{n_i}\| \leq (1 - t_{n_i})(\|f(x_0)\| + \alpha\|x_{n_i} - x_0\|) + t_{n_i}\|x_0\|.$$

Hence by dividing $\|x_{n_i} - x_0\|$ and limiting n_i to ∞, we have a contradiction that $1 \leq \alpha$. Since X is reflexive and (x_{n_i}) is bounded, a sequence

(x_{n_i}) has a subsequence which converges weakly to $y_0 \in C$. Since $f(x_{n_i})$ is also bounded and

$$\|x_{n_i} - f(x_{n_i})\| = t_{n_i}\|x_0 - f(x_{n_i})\| \longrightarrow 0$$

and $I - f$ is demi-closed, y_0 is a fixed point of f. □

With the following definition we also have a fixed point theorem.

DEFINITION 5. Let C be a subset of a Banach space X. We say that $f : C \to X$ is asymptotically *c'*-contractive on C if there is a function $c : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$such that

(1) c is symmetric (i.e., for all $s, t \in \mathbb{R}^+$, $c(s,t) = c(t,s)$)
(2) for all $s \in \mathbb{R}^+$,

$$\limsup_{t\to\infty} c(s,t) < 1$$

(3) for all $s, t \in \mathbb{R}^+$, $c(s,t) \le 1$
(4) for all $x, y \in C$,

$$\|f(x) - f(y)\| \le c(\|x\|, \|y\|)\|x - y\|.$$

THEOREM 6. *Let Xbe a reflexive Banach space and let Cbe a unbounded closed convex subset of X. Let $f : C \to X$ be asymptotically c'-contractive on C such that $f(C) \subset C$ and $I - f$ is demi-closed. Then f has a fixed point.*

Proof. The proof is similar with the proof of Theorem 4. □

EXAMPLE 1. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f(x) = \begin{cases} 2x + 1, & 0 \le x \le 1, \\ \dfrac{1}{2}x + \dfrac{5}{2}, & 1 < x. \end{cases}$$

Then f is asymptotically c-contractive on $\mathbb{R}^+$ with c defined by

$$c(s,t) = \begin{cases} \frac{1}{2}, & 1 \le s, t, \\ \frac{1}{2} + \frac{3}{2t}, & 0 \le s < 1, 1 < t \\ \frac{1}{2} + \frac{3}{2s}, & 0 \le t < 1, 1 < s \\ 2, & 0 \le t < 1, 0 \le s < 1 \end{cases}$$

and $r_0 = 1, r_1 = 3$. Note that f is not nonexpansive. Of course we can apply the Picard-Banach contraction principle to this Example directly with $C = [1, \infty)$. But in Theorem 4 we note that if X has a dimension more than 1 , $C - r_0 B_X$ is not necessarily convex and f is not necessarily nonexpansive. In conclusion we obtain a fixed point theorem for a mapping where its behavior at infinity is nonexpansive and contraction radially.

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