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THE ZEROS OF SOLUTIONS OF SOME DIFFERENTIAL INEQUALITIES

RakJoong Kim

Abstract. Let $x(t)$ satisty

$$\big(p(t)x'(t)\big)' + q(t)x^{\alpha}(t) + r(t)x^{\beta-1}x'(t) \le 0 \ (\ge 0).$$

Then the zeros of $x(t)$ or $x'(t)$ are simple.

1. Introduction

This paper is concerned with zeros of solutions to the inequality of the following type: For $\alpha \ge 1,\ \beta \ge 1$,

$$x(t)\left\{(p(t)x'(t))' + q(t)x^{\alpha}(t) + r(t)x^{\beta-1}(t)x'(t)\right\} \le 0. \tag{1.1}$$

By methods of variation of constants Kwong[1] proved that $y'(a) \ne 0$ or $y'(b) \ne 0$ if $y(t)$ is positive(negative) in (a, b) and if $y(t)$ satisfies the inequality

$$y''(t) + f(t)y'(t) + g(t)y(t) \le (\ge)0.$$

with $y(a) = 0$ or $y(b) = 0$ where f and g are continuous functions. Using LaSalle's inequality Wong[2] proved the same results for inequality of the type:

$$(p(t)x'(t))' + g(t)F(x(t)) \le (\ge)0.$$

We consider a simple case: Let $\phi(t)$ be positive, nondecreasing and $\int_0^r 1/\phi(s)\, ds = \infty$ for any fixed $r > 0$ and let $p(t)$ be positive for $t \ge a$

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and $|q(t)/p(t)|$ integrable on the any compact interval. Suppose that $x(t)$ satisfies the inequality

$$p(t)x'(t) \pm q(t)\phi(x(t)) \leq 0, \qquad t \in (a,b)$$

Let $x(t) > 0$ in $(a,t]$. Then $x(a) \neq 0$. Otherwise, then

$$\int_0^{x(t)} \frac{ds}{\phi(s)} \leq \int_a^t \left|\frac{q(s)}{p(s)}\right| \, ds.$$

Thus $x(t) \equiv 0$ in $[a,t]$. Let $x(t)$ satisfy the nonlinear differential equation $p(t)x'(t) \pm q(t)\phi(x(t)) = 0$. If $\phi(0) = 0$, $p(a) \neq 0$ and $x(a) = 0$, then $x(t)$ is flat at $t = a$. i.e., $x^{(n)}(a) = 0$ for all $n \in \mathbb{N}$.

2. main Results

In order to prove our main results, we need the following integral inequality called LaSalle's inequality.

LaSalle's Inequality. *For some $c > 0$ let*

(C1) $F \in C([0,c];[0,\infty))$ *be positive and nondecreasing on* $(0,c)$,

(C2) $$h \in L_1(\mathbb{R};[0,\infty)),$$

(C3) $$x \in C([a,b];[0,c)).$$

Then for $t \in [a,b]$ the inequalities

$$x(t) \leq \int_a^t h(s)F(x(s)) \, ds, \tag{2.1}$$

$$x(t) \leq \int_t^b h(s)F(x(s)) \, ds \tag{2.2}$$

imply that

$$\int_0^{x(t)} \frac{ds}{F(s)} \leq \int_a^t h(s) \, ds, \tag{2.3}$$

$$\int_0^{x(t)} \frac{ds}{F(s)} \leq \int_t^b h(s) \, ds \tag{2.4}$$

respectively. In addition ,

$$\text{(C4)} \qquad \textit{if } \int_0^{\epsilon} \frac{ds}{F(s)} \textit{ is divergent for } \epsilon > 0 \textit{ then } x(t) \equiv 0 \textit{ on } [a,b]$$

Throughout this paper we suppose that $1/p(t)$ is positive for $t \in (a,b]$ and integrable in $[a,b]$.

THEOREM 1. *Assume that*

$$|q(t)| \leq h(t), \tag{2.5}$$

$$\frac{|r(t)|}{p(t)} \leq h(t), |r'(t)| \leq h(t) \textit{ in } (a,b) \tag{2.6}$$

$$h(t) \in L_1([a,b];(0,\infty)). \tag{2.7}$$

Let $x(t)$ satisfy (1.1). *Assume that $x(t)$ be either positive or negative in (a,b). Then*

$$x^2(a) + (x')^2(a) \neq 0, \tag{2.8}$$

or

$$x^2(b) + (x')^2(b) \neq 0. \tag{2.9}$$

Proof. We prove (2.8). Suppose that $x^2(a) + (x')^2(a) = 0$.
Case (1): Assume that $x(t) > 0$ in (a,b). It follows that

$$p(t)x'(t) \leq -\int_a^t \left\{q(s)x^{\alpha}(s) + r(s)x^{\beta-1}(s)x'(s)\right\} ds$$

which implies

$$p(t)x'(t) \leq \frac{1}{\beta}\left[-r(t)x^{\beta}(t) - \int_a^t \left\{\beta q(s)x^{\alpha}(s) - r'(s)x^{\beta}(s)\right\} ds\right].$$

Thus we have

$$
\begin{aligned}
&x(t)\\
&\le \frac{1}{\beta}\left[\int_a^t -\frac{r(s)}{p(s)}x^\beta(s)ds - \int_a^t \frac{1}{p(s)}\int_a^s \left\{\beta q(\tau)x^\alpha(\tau) - r'(\tau)x^\beta(\tau)\right\} d\tau ds\right]\\
&\le \frac{1}{\beta}\left[\int_a^t \frac{|r(s)|}{p(s)}x^\beta(s)ds + K\int_a^t \left\{\beta|q(s)|x^\alpha(s) + |r'(s)|x^\beta(s)\right\} ds\right]\\
&= \int_a^t \left[K|q(s)|x^\alpha(s) + \frac{1}{\beta}\left\{\frac{|r(s|)}{p(s)} + K|r'(s)|\right\} x^\beta(s)\right] ds\\
&\le \int_a^t (2K+1)h(s)\{x^\alpha(s) + x^\beta(s)\}ds,
\end{aligned}
$$

where $K = \int_a^b ds/p(s)\ ds$. Since $F(s) = s^\alpha + s^\beta$ is increasing in $s > 0$ by means of (2.1), (2.3) we obtain

$$
\int_0^{x(t)} \frac{ds}{s^\alpha + s^\beta} \le (2K+1)\int_a^t h(s)\ ds.
$$

Consequently we obtain $x(t) \equiv 0$ in $[a,t]$. This contradicts the hypothesis $x(t) > 0$ in (a,b).
Case(2): Assume that $x(t) < 0$ in (a,b). It follows that

$$
p(t)x'(t) \ge -\int_a^t \left\{q(s)x^\alpha(s) + r(s)x^{\beta-1}(s)x'(s)\right\}\ ds
$$

which implies

$$
p(t)x'(t) \ge \frac{1}{\beta}\left[-r(t)x^\beta(t) - \int_a^t \left\{\beta q(s)x^\alpha(s) - r'(s)x^\beta(s)\right\} ds\right].
$$

Thus we have

$$
\begin{aligned}
x(t) \ge \frac{1}{\beta}\int_a^t -\frac{r(s)}{p(s)}x^\beta(s)\ ds&\\
-\int_a^t \frac{1}{p(s)}\int_a^s &\left\{q(\tau)x^\alpha(\tau) - \frac{1}{\beta}r'(\tau)x^\beta(\tau)\right\} d\tau ds.
\end{aligned}
$$

which is reduced to

$$|x(t)| \leq \frac{1}{\beta}\int_a^t \frac{|r(s)|}{p(s)}|x(s)|^\beta ds + \int_a^t \frac{1}{p(s)}\int_a^s \left\{|q(\tau)||x(\tau)|^\alpha + \frac{1}{\beta}|r'(\tau)||x(\tau)|^\beta\right\} d\tau ds.$$

Therefore we have

$$\begin{aligned} |x(t)| &\leq \int_a^t \left[K|q(s)||x(s)|^\alpha + \frac{1}{\beta}\left\{\frac{|r(s)|}{p(s)} + K|r'(s)|\right\}|x(s)|^\beta\right] ds \\ &\leq \int_a^t (2K+1)h(s)\{|x(s)|^\alpha + |x(s)|^\beta\}ds, \end{aligned}$$

where $K = \int_a^b ds/p(s)\ ds$. By means of (2.1), (2.3) we obtain

$$\int_0^{|x(t)|} \frac{ds}{s^\alpha + s^\beta} \leq (2K+1)\int_a^t h(s)\ ds.$$

Consequently we obtain $x(t) \equiv 0$ in $[a,t]$. This contradicts the hypothesis $x(t) < 0$ in (a,b).

Next we prove (2.9). Let $x(t) > 0$ in (a,b). In the case $x(t) < 0$ in (a,b) we can apply the similar method. It follows that

$$-p(t)x'(t) \leq \frac{1}{\beta}\left[r(t)x^\beta(t) - \int_t^b \left\{\beta q(s)x^\alpha(s) - r'(s)x^\beta(s)\right\} ds\right],$$

Thus we obtain

$$\begin{aligned} x(t) &\leq \frac{1}{\beta}\left[\int_t^b \frac{|r(s)|}{p(s)}x^\beta(s)ds + K\int_t^b \left\{\beta|q(s)|x^\alpha(s) + |r'(s)|x^\beta(s)\right\} ds\right] \\ &= \int_t^b \left[K|q(s)|x^\alpha(s) + \frac{1}{\beta}\left\{\frac{|r(s|)}{p(s)} + K|r'(s)|\right\}x^\beta(s)\right] ds \\ &\leq \int_t^b (2K+1)h(s)\{x^\alpha(s) + x^\beta(s)\}ds, \end{aligned}$$

where $K = \int_a^b ds/p(s)\ ds$ provided that $x(t) > 0$. Thus by means of (2.2), (2.4) we reach the same conclusion as case (1). □

Corollary 2. *Assume that $x(t)$ satisfies*

$$x(t)\left\{(p(t)x'(t))' + q(t)x^{\alpha}(t) + r(t)x^{\beta}(t)\right\} \le 0 \tag{2.10}$$

where $\alpha \ge 1,\ \beta \ge 1$ with the conditions

$$|q(t)| \le h(t),\ |r(t)| \le h(t) \quad \text{in } (a,b) \tag{2.11}$$

$$h(t) \in L_1([a,b];(0,\infty)). \tag{2.12}$$

Let $x(t)$ be either positive or negative in (a,b). Then $x^2(a)+(x')^2(a) \neq 0$ or $x^2(b)+(x')^2(b) \neq 0$.

We consider the following singular differential inequality .

Example 3. For $\sigma > -1$ let $x(t)$ satisfy

$$(t^{\sigma}x'(t))' + q(t)x^{\alpha}(t) + x^5(t) \le 0, \qquad \alpha \ge 1.$$

with condition $|q(t)| \in L_1([0,1];(0,\infty))$. Let $x(t)$ be positive in $(0,1)$. If $x(0)=0$ or $x(1)=0$ Then $x'(0) \neq 0$ or $x'(1) \neq 0$.

Theorem 4. *Under the assumptions of Theorem 1 let $x(t)$ satisfy (1.1). Unless $x(t)$ is constant $x(t)$ has only finitely many zeros in every compact interval $[a,b]$.*

Proof. Assume that $E = \{t \in [a,b] | x(t) = 0\}$ is an infinite set. Then E contains a sequence $\{t_n\}_{n\in\mathbb{N}}$ convergent in E, say t_0. Then $x'(t_0)=0$. This contradicts Theorem 1 because $x(t_0) = x'(t_0) = 0$. □

Theorem 5. *Under the assumptions of Theorem 1 with $q(t) \neq 0$ if $x(t)$ is a nontrivial solution of*

$$(p(t)x'(t))' + q(t)x^{\alpha}(t) + r(t)x^{\beta-1}(t)x'(t) = 0,$$

then $x'(t)$ has only a finite number of zeros in every compact interval $[a,b]$.

Proof. Assume that $F = \{t \in [a,b] \mid x'(t) = 0\}$ is an infinite set. Then F contains a sequence $\{t_n\}_{n\in\mathbb{N}}$ convergent to some element in F. Call it t_0. Then $x''(t_0)=0$. So $x(t_0)=0$. This contradicts Theorem 1 □

Theorem 6. *Under the assumption in Theorem 1 with* $q(t) > 0$ *and* $r(t) > 0$ *let* $x(t)$ *be a nontrivial* C^1*-solution of*

$$x(t)\left\{(p(t)x'(t))' + q(t)x^{\alpha}(t) + r(t)x^{\beta-1}(t)x'(t)\right\} \le 0$$

where $\alpha(\ge 1)$ *is an odd integer and* $\beta(\ge 2)$ *is an even integer. Then* $x(t)$ *has no nonnegative local minimum or nonpositive local maximum.*

Proof. From Theorem 1 it follows that 0 is neither a local maximum nor a local minimum. Let $x(s)$ be a positive local minimum. Then $x'(s) = 0$. We put $y(t) = x(t) - x(s)$. Then we find $y(s) = y'(s) = 0,\ y'(t) = x'(t)$. There exists $\delta > 0$ such that $y(t) > 0,\ y'(t) \ge 0$ for $t \in (s, s+\delta)$. So $x(t) > y(t)$ for $t \in (s, s+\delta)$. Since for $t \in (s, s+\delta)$

$$\begin{aligned} &(p(t)y'(t))' + q(t)y^{\alpha}(t) + r(t)y^{\beta-1}(t)y'(t) \\ &= (p(t)x'(t))' + q(t)y^{\alpha}(t) + r(t)y^{\beta-1}(t)x'(t) \\ &\le (p(t)x'(t))' + q(t)x^{\alpha}(t) + r(t)x^{\beta-1}(t)x'(t), \end{aligned}$$

we obtain

$$y(t)\left\{(p(t)y'(t))' + q(t)y^{\alpha}(t) + r(t)y^{\beta-1}(t)y'(t)\right\} \le 0.$$

But this contradicts Theorem 1. In the case $x(t) < 0$ in (a,b) we can apply the similar method. □

Now we introduce a function $\phi(t) \in C([a,\infty);[a,\infty))$ which is nondecreasing with $\phi(t) \le t$ on $[a,\infty)$.

Theorem 7. *Assume that*

(C5) $1/p(t) \in L_1([a,b];(0,\infty)),\ q(t) \in L_1([a,b];\mathbb{R}),$

(C6) $G \in L_1(\mathbb{R},\mathbb{R})$ *and there exists a function* F *satisfying* $(C1),\ (C4)$ *and* $|G(x)| \le F(|x|)$ *in* $x \in (-\epsilon,\epsilon)$ *for some* $\epsilon > 0$.

Suppose that

$$x(t)\left\{(p(t)x'(t))' \pm q(t)G(x(\phi(t)))\right\} \le 0. \tag{2.13}$$

Let $x(t)$ *satisfy* (2.13). *Assume that* $x(t)$ *is either positive or negative in* (a,b). *Then* $x^2(a) + (x')^2(a) \ne 0$ *or* $x^2(b) + (x')^2(b) \ne 0$.

Proof. We prove only the case $x(t) < 0$ in (a, b). Direct calculation leads to

$$|x(t)| \leq K \int_a^t |q(s)| \ F(|x(\phi(s))|)ds.$$

where $K = \int_a^t ds/p(s) \ ds$. Put $X(t) = K \int_a^t |q(s)| \ F(|x(\phi(s))|)$. Then We have

$$\begin{aligned} X'(t) &= K|q(t)| \ F(|x(\phi(t))|) \\ &\leq K|q(t)| \ F(X(\phi(t))) \\ &\leq K|q(t)| \ F(X(t)). \end{aligned}$$

We note that $X(t)$ is increasing. Thus we obtain

$$\int_0^{|x(t)|} \frac{du}{F(u)} \leq K \int_a^t |q(s)| \ ds.$$

By (C4) $x(t) \equiv 0$ in $[a, t]$. This contradicts the hypothesis $x(t)$ is negative in (a, b). □

THEOREM 8. *With the assumptions (C6) and*

$$1/p(t) \in L_1([a, b]; (0, \infty)), \quad q \circ \phi \in L_1([a, b]; \mathbb{R})$$

we suppose that

$$x(t) \left\{ (p(t)x'(t))' \pm q(\phi(t))G(x(\phi(t)))\phi'(t) \right\} \leq 0. \tag{2.14}$$

Let $x(t)$ satisfy (2.13). Assume that $x(t)$ is either positive or negative in (a, b). Then $x^2(a) + (x')^2(a) \neq 0$ or $x^2(b) + (x')^2(b) \neq 0$.

Proof. It follows that $\phi'(t) \geq 0$ because $\phi(t)$ is nondecreasing . Suppose that $x^2(a) + (x')^2(a) = 0$. If $x(t) < 0$ in (a,b) by the same process as in above we obtain

$$\int_0^{|x(t)|} \frac{du}{F(u)} \leq K \int_{\phi(a)}^{\phi(t)} |q(s)| \ ds.$$

By (C4) $x(t) \equiv 0$ in $[a, t]$. This contradicts the hypothesis that $x(t)$ is negative in (a, b). □

EXAMPLE 9. For $\alpha \geq 1, \ t \geq 1$ let $x(t)$ satisfy

$$(p(t)x'(t))' + q(t)x^{\alpha}(\sqrt{t}) \leq 0.$$

with condition $|q(t)| \in L_1([a, b]; (0, \infty)), \ 1 < a < b$. Let $x(t)$ be positive in (a, b). If $x(a) = 0$ or $x(b) = 0$ then $x'(a) \neq 0$ or $x'(b) \neq 0$.

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Department of Mathematics
Hallym University
Chuncheon, Kangwon 200-702, Korea.
E-mail: rjkim@hallym.ac.kr