ON RIGHT(LEFT) DUO PO-SEMIGROUPS

S. K. LEE AND K. Y. PARK

Abstract. We investigate some properties on right(resp. left) duo po-semigroups.

1. Introduction

Kehayopulu([6]) prove that every ideal of an \( \mathcal{N} \)-class of an ordered semigroup does not contain proper prime ideals. As a consequence, each prime ideal of an ordered semigroup is decomposable into its \( \mathcal{N} \)-classes.

In this paper, we give the relation between the left(resp. right) filters and the prime left(resp. right) ideals. We define a semilattice congruence \( \mathcal{N}_l \)(resp. \( \mathcal{N}_r \)) generated by the left(resp. right) filter on a right(resp. left) duo po-semigroup and investigate some properties on the right(resp. left) duo po-semigroups. Also we prove that every left(resp. right) ideal of \( \mathcal{N}_l \)-class resp. \( \mathcal{N}_r \)-class of a right(resp. left) duo po-semigroup does not contain the proper prime left(resp. right) ideals. As a consequence, each prime left(resp. right) ideal of a right(resp. left) duo po-semigroup is decomposable into its \( \mathcal{N}_l \)-classes(resp. \( \mathcal{N}_r \)-classes).

A po-semigroup (: ordered semigroup) is an ordered set \( S \) at the same time a semigroup such that \( a \leq b \Rightarrow xa \leq xb \) and \( ax \leq bx \) for all \( x \in S \).

Let \( S \) be a po-semigroup. A nonempty subset \( A \) of \( S \) is called a left(resp. right) ideal of \( S \) if (1) \( SA \subseteq A \)(resp. \( SA \subseteq A \)), (2) \( a \in A \) and \( b \leq a \) for \( b \in S \) \( \Rightarrow b \in A \([3,5]\)) \( A \) is called an ideal of \( S \) if it is a left and right ideal of \( S \).

Received July 14, 2003.
2000 Mathematics Subject Classification: 03G25, 06F35.
Key words and phrases: po-semigroup, ideal, left(right) ideal, right(left) duo, prime, prime left(right) ideal, left(right) filter, left(right) congruence, congruence, semilattice congruence.
A po-semigroup $S$ is said to be right (resp. left) duo if every right (resp. left) ideal is a left (resp. right) ideal ([4,5]).

A non-empty subset $T$ of a po-semigroup $S$ is said to be prime if $AB \subseteq T \implies A \subseteq T$ or $B \subseteq T$ for subsets $A, B$ of $S$ ([8]). Equivalent Definition: For elements $a, b$ in a subset $T$ if $ab \in T \Rightarrow a \in T$ or $b \in T$. $T$ is called a prime left (resp. right) ideal if $T$ is prime as a left (resp. right) ideal ([2]).

A non-empty subsemigroup $F$ of a po-semigroup $S$ is called a left (resp. right) filter of $S$ if $ab \in F$ for $a, b \in S \implies b \in F$ (resp. $a \in F$) if $F$ is a left and right filter ([2,4,5]).

An equivalence relation $\sigma$ on $S$ is called a left congruence (resp. right congruence) on $S$ if $(a, b) \in \sigma \implies (ac, bc) \in \sigma$ (resp. $(ca, cb) \in \sigma$) for all $c \in S$. An equivalence relation $\sigma$ on $S$ is called a congruence if it is a left and right congruence. A relation $\sigma$ is called a semilattice congruence on $S$ if $\sigma$ is a congruence such that $(x^2, x) \in \sigma$ and $(xy, yx) \in \sigma([1,2,4])$.

**Notation.** For a semilattice congruence $\sigma$, $(z)_{\sigma}$ is a class of the semilattice congruence $\sigma$ containing an element $z$ in a po-semigroup $S$.

## 2. Main Results.

**Lemma ([9]).** Let $S$ be a po-semigroup and $F$ a nonempty subset of $S$. The following are equivalent:
1) $F$ is a left (resp. right) filter of $S$.
2) $S \setminus F = \emptyset$ or $S \setminus F$ is a prime left (resp. right) ideal of $S$.

From Lemma, we get the following corollary.

**Corollary 1([2]).** Let $S$ be a po-semigroup and $F$ a nonempty subset of $S$. The following are equivalent:
1) $F$ is a filter of $S$.
2) $S \setminus F = \emptyset$ or $S \setminus F$ is a prime ideal of $S$.

**Proposition 1.** A po-semigroup $S$ does not contain proper left (resp. right) filters if and only if $S$ does not contain proper prime left (resp. right) ideals.

**Proof.** $\Rightarrow$. Assume that $S$ contains a proper prime left ideal $L$ of $S$. Then $\emptyset \neq S \setminus L \subseteq S$. Since $S \setminus (S \setminus L) = L$, we note that $S \setminus (S \setminus L)$
is a prime left ideal of $S$. By Lemma 1, $S \setminus L$ is a proper left filter of $S$. It is impossible. Hence $S$ does not contain proper prime left ideals. 

$\leftarrow$. Suppose that $F$ is a proper left filter of $S$. Then $S \setminus F \neq \emptyset$. By Lemma 1, $S \setminus F$ is a proper prime left ideal of $S$. It is impossible. Hence $S$ does not contain proper prime left filters. 

By Proposition 1, we have the following corollary.

**Corollary 2** ([6, Remark 2]). A po-semigroup $S$ does not contain proper filters if and only if $S$ does not contain proper prime ideals.

Now we define a relation “$N_i$” on a po-semigroup $S$ as follows:

$$N_i := \{(x, y) | N_i(x) = N_i(y)\}, \quad N_r := \{(x, y) | N_r(x) = N_r(y)\}$$

where $N_i(x)$ (resp. $N_r(x)$) is the left (resp. right) filter of $S$ generated by $x \in S$.

**Proposition 2.** $N_i$ (resp. $N_r$) is a semilattice congruence on a right (resp. left) duo po-semigroup $S$.

**Proof.** It is easy to check that $N_i$ is an equivalence relation on $S$.

Let $(x, y) \in N_i$. Then $N_i(x) = N_i(y)$. Since $xz \in N_i(xz)$ for all $z \in S$ and $N_i(xz)$ is a left filter, we get $x \in N_i(xz)$ and $z \in N_i(xz)$. Thus $N_i(x) \subseteq N_i(xz)$ and so $y \in N_i(y) = N_i(x) \subseteq N_i(xz)$. Since $y, z \in N_i(xz)$ and $N_i(xz)$ is a subsemigroup of $S$, we get $yz \in N_i(xz)$. Therefore $N_i(yz) \subseteq N_i(xz)$. By symmetry, we get $N_i(xz) \subseteq N_i(yz)$. Hence $N_i(xz) = N_i(yz)$. Therefore $N_i$ is a right congruence.

Now we shall show that $(x^2, x) \in N_i$. Let $x \in S$. Since $x^2 \in N_i(x^2)$ and $N_i(x^2)$ is a left filter, we get $x \in N_i(x^2)$. Thus $N_i(x) \subseteq N_i(x^2)$. Since $x \in N_i(x)$ and $N_i(x)$ is a subsemigroup of $S$, we get $x^2 \in N_i(x)$. Hence $N_i(x^2) \subseteq N_i(x)$. Therefore $N_i(x^2) = N_i(x)$, and so $(x^2, x) \in N_i$.

Next we shall show that $(xy, yx) \in N_i$. Let $x, y \in S$. Since $xy \in N_i(xy)$ and $N_i(xy)$ is a left filter, we have $x \in N_i(xy)$. Suppose that $y \notin N_i(xy)$. Then $y \in S \setminus N_i(xy)$. Since $S \setminus N_i(xy)$ is a prime right ideal and $S$ is a right duo, $xy \in S(S \setminus N_i(xy)) \subseteq S \setminus N_i(xy)$. It is impossible. Thus $y \in N_i(xy)$. Since $N_i(xy)$ is a filter, $yx \in N_i(xy)$. Thus $N_i(yx) \subseteq N_i(xy)$. By symmetry, $N_i(xy) \subseteq N_i(yx)$. Therefore $N_i(xy) = N_i(yx)$ and so $(xy, yx) \in N_i$. 

On right(left) duo po-semigroups
Finally, we shall show that $\mathcal{N}_i$ is a left congruence. Let $(x, y) \in \mathcal{N}_i$, and $z \in S$. Then $\mathcal{N}_i(xz) = \mathcal{N}_i(xz) = \mathcal{N}_i(yz) = \mathcal{N}_i(zy)$. Therefore $\mathcal{N}_i$ is a left congruence. It follows that $\mathcal{N}_i$ is a semilattice congruence.

**Proposition 3.** Let $S$ be a po-semigroup. If $F$ is a left filter of $S$ and $F \cap (z)_{\mathcal{N}_i} \neq \emptyset$ for $z \in S$, then $(z)_{\mathcal{N}_i} \subseteq F$.

*Proof.* Assume that $F$ is a left filter of $S$ and $a \in F \cap (z)_{\mathcal{N}_i}$ for $z \in S$. If $y \in (z)_{\mathcal{N}_i}$ then $(y)_{\mathcal{N}_i} = (z)_{\mathcal{N}_i} = (a)_{\mathcal{N}_i}$. Thus $(y, a) \in \mathcal{N}_i$, and so $\mathcal{N}_i(y) = \mathcal{N}_i(a)$. Since $F$ is a left filter of $S$ and $a \in F$, we have $\mathcal{N}_i(a) \subseteq F$. Thus $y \in \mathcal{N}_i(y) = \mathcal{N}_i(a) \subseteq F$. Hence $(z)_{\mathcal{N}_i} \subseteq F$. □

**Proposition 4.** For a po-semigroup $S$, $a \leq b$ implies $(a, ba) \in \mathcal{N}_i$ and $(a, ab) \in \mathcal{N}_r$.

*Proof.* Suppose that $a \leq b$. Since $a \in N_i(a)$ and $N_i(a)$ is a left filter, we get $b \in N_i(a)$. Thus $ba \in N_i(a)$, and so $N_i(ba) \subseteq N_i(a)$. Since $ba \in N_i(ba)$ and $N_i(ba)$ is a left filter, we have $a \in N_i(ba)$. Thus $N_i(a) \subseteq N_i(ba)$. Hence $N_i(a) = N_i(ba)$, and so $(a, ba) \in \mathcal{N}_i$.

By symmetry, we can prove that $(a, ab) \in \mathcal{N}_r$. □

**Proposition 5.** Let $S$ be a right duo po-semigroup. If $L$ is a left ideal of $(z)_{\mathcal{N}_i}$ for $z \in S$ then $L$ does not contain proper prime left ideals.

*Proof.* From Proposition 1, it is sufficient to prove that $L$ does not contain proper left filters (of $L$). Let $F$ be a left filter of $L$ and $a \in F$. Now we define $T := \{x \in S \mid a^2x \in F\}$. Then $T$ is a nonempty set, since $a^2a = a^3 \in F$.

Now we show that $F = T \cap L$. If $y \in F$, then $a^2y \in F$. Thus $y \in T$. Since $F$ is a left filter of $L$, $F \subseteq L$. Hence $y \in T \cap L$, and so $F \subseteq T \cap L$. Conversely, if $y \in T \cap L$, then $a^2y \in F$. Since $F$ is a left filter of $L$, we get $y \in F$. Therefore $F = T \cap L$.

Next we show that $T$ is a left filter of $L$. If $x \in T$ and $y \in T$, then $a^2x, a^2y \in F$. Since $F$ is a left filter, we have $x, y \in F$. Since $a \in F$, $a^2xy \in F$. Thus $xy \in T$. If $xy \in T$ for $x, y \in L$, then $(a^2x)y = a^2(xy) \in F$. Since $F$ is a left filter of $L$, we get $y \in F$. If $x \in T$ and $x \leq y$ for $y \in L$, then $a^2x \in F$. Since $x \leq y$, we get $a^2x \leq a^2y$. Since $F$ is a left filter, $a^2y \in F$. Thus $y \in T$. Therefore $T$ is a left filter of $L$. 
We note that \( a \in F = T \cap L \subseteq L \subseteq (z)_{N_t} \), and so \( T \cap (z)_{N_t} \neq \emptyset \). Since \( T \) is a left filter of \( L \), we have \( (z)_{N_t} \subseteq T \) by Proposition 3. Thus \( L = (z)_{N_t} \cap L \subseteq T \cap L = F \subseteq L \), and so \( F = L \). Hence \( L \) does not contain proper left filters (of \( L \)). Therefore by Proposition 1, \( L \) does not contain proper prime right ideals. \( \square \)

**Proposition 6.** Let \( S \) be a right duo po-semigroup and \( L \) a prime left ideal of \( S \). Then \( L = \cup \{(x)_{N_t} \mid x \in L\} \).

*Proof.* Let \( t \in (x)_{N_t} \) for some \( x \in L \). Since \( (x)_{N_t} \) is a left ideal of \( (x)_{N} \), \( (x)_{N_t} \) does not contain proper prime left ideals by Proposition 5. If we prove that \( (x)_{N_t} \cap L \) is a prime left ideal of \( (x)_{N_t} \) then \( (x)_{N_t} \cap L = (x)_{N_t} \).

We first show that \( (x)_{N_t} \cap L \) is a left ideal of \( (x)_{N_t} \). We note that \( (x)_{N_t} \cap L \neq \emptyset \) since \( x \in (x)_{N_t} \cap L \). And \( (x)_{N_t}((x)_{N_t} \cap L) = (x)_{N_t}^2 \cap (x)_{N_t}L \subseteq (x)_{N_t} \cap SL \subseteq (x)_{N_t} \cap L \). Let \( a \in (x)_{N_t} \cap L \) and \( b \leq a \) for \( b \in (x)_{N_t} \). Since \( L \) is a left ideal of \( S \), \( b \) is contained in \( L \). Thus \( b \in (x)_{N_t} \cap L \). Hence \( (x)_{N_t} \cap L \) is a left ideal of \( (x)_{N_t} \).

Finally, we show that \( (x)_{N_t} \cap L \) is prime in \( (x)_{N_t} \). Let \( yz \in (x)_{N_t} \cap L \) for \( y, z \in (x)_{N_t} \). Since \( yz \in L \) and \( L \) is a prime left ideal of \( S \), \( y \) is contained in \( L \) or \( z \) is contained in \( L \). Hence \( y \in (x)_{N_t} \cap L \) or \( z \in (x)_{N_t} \cap L \). Therefore \( (x)_{N_t} \cap L \) is a prime left ideal of \( (x)_{N_t} \).

It follows that

\[
L \subseteq \cup \{(x)_{N_t} \mid x \in L = \cup \{(x)_{N_t} \cap L \mid x \in L \} \subseteq L.
\]

Therefore \( L = \cup \{(x)_{N_t} \mid x \in L\} \). \( \square \)

B. by similar methods of Proposition 3, 5 and 6, we have the followings:

(1) If \( F \) is a right filter of a po-semigroup \( S \) and \( F \cap (z)_{N_r} \neq \emptyset \) for \( z \in S \), then \( (z)_{N_r} \subseteq F \).

(2) If \( R \) is a right ideal of \( (z)_{N_r} \) of left duo po-semigroups then \( R \) does not contain proper prime right ideals.

(3) If \( R \) is a prime right ideal of left duo po-semigroups, then

\[
R = \cup \{(x)_{N_r} \mid x \in R\}.
\]
3. Examples

Now we give an example of a left filter which is not a right filter in po-semigroups and an example of a left and right filter in a po-semigroup.

**Example 1**([7]). Let $S := \{a, b, c, d, e, f\}$ be a po-semigroup with Cayley table and Hasse diagram on $S$ as follows:

<table>
<thead>
<tr>
<th>·</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>e</td>
<td>e</td>
<td>e</td>
<td>e</td>
<td>e</td>
<td>e</td>
<td>e</td>
</tr>
<tr>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
</tr>
</tbody>
</table>

The set $A := \{e, f\}$ is a left filter, but not a right filter of $S$. Thus $A$ is not a filter of $S$.

**Example 2**([8]). Let $S := \{a, b, c, d, e, f\}$ be a po-semigroup with Cayley table (Table 2) and Hasse diagram (Figure 2) on $S$ as follows:

<table>
<thead>
<tr>
<th>·</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>d</td>
<td>e</td>
<td>f</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>b</td>
<td>b</td>
<td>d</td>
<td>e</td>
<td>f</td>
</tr>
<tr>
<td>e</td>
<td>e</td>
<td>f</td>
<td>f</td>
<td>e</td>
<td>e</td>
<td>f</td>
</tr>
<tr>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
</tr>
</tbody>
</table>

Table 2

The set $B := \{a, d, e\}$ is a left and right filter of $S$, and so $B$ is a filter of $S$. 
References


Department of Mathematics Education
Gyeongsang National University
Chinju 660-701, Korea
E-mail: sklee@nongae.gsnu.ac.kr