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LIMIT THEOREM OF THE DOUBLY STOCHASTIC MATRICES

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Abstract. In this paper we shall study the limit of the doubly stochastic matrices obtained from Boolean regular matrices.

1. Introduction

In this paper we shall explain how our Boolean regular matrices lead to doubly stochastic matrices. Suppose that Ω is a $\sigma-$algebra on a non-empty set S. Then we can give a probability measure P on Ω as a function of elements of Ω satisfying three axioms :

i) For every element $a_{ij} \in \Omega$, the value of the function is a non-negative number : $P(a_{ij}) \geq 0$.
ii) For any two disjoint elements a_{ij} and a_{ik}, the value of the function for their union $a_{ij} \cup a_{ik}$ is equal to the sum of its value for a_{ij} and its value for a_{ik} : $P(a_{ij} \cup a_{ik}) = P(a_{ij}) + P(a_{ik})$ provided $a_{ij} \cap a_{ik} = \emptyset$.
iii) The value of the function for S is equal to 1 : $P(S) = 1$. [Ref : p.23.[2]]

Let $A = (a_{ij}) \in M_n(2^s)$ be a Boolean regular matrix (That is, $\sum_{k=1}^{n} a_{ik} = s = \sum_{k=1}^{n} a_{ki}, a_{ik} \cap a_{jk} = \emptyset = a_{ki} \cap a_{kj}$ for $i \neq j$.). Then by the properties of the Boolean regular matrix we obtain a doubly stochastic matrix $P = (p_{ij})$ such that $p_{ij} = P(a_{ij})$.

In this paper we use the Definitions and a Theorem which is related to the stochastic matrix and a fixed vector in [5].

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Definition 1.1. A row vector $u = (u_1, u_2, \cdots, u_n)$ is called a probability vector if its components are nonnegative and their sum is 1.

Definition 1.2. A square matrix $A = (a_{ij})$ is called a stochastic matrix if each of its rows is a probability vector, i.e. if each entry of A is non-negative and the sum of the entries in every row is 1.

Definition 1.3. A stochastic matrix $A = (a_{ij})$ is called a semi-positive if all the entries of some power A^m are positive.

Definition 1.4. A non-zero row vector $u = (u_1, u_2, \cdots, u_n)$ is called a fixed point of a square matrix A if u is left fixed, i.e. is not changed, when multiplied by A :

$$uA = u \ .$$

NOTE : If u is a fixed vector of a matrix A, then for any real number $r \neq 0$ the scalar multiple ru is also a fixed vector of A.

Theorem 1.5. *Let A be a semi-positive stochastic matrix. Then :*

(1) *A has a unique fixed probability vector t, and the components of t are all positive ;*
(2) *the sequence $A, A^2, A^3, \cdots$ of powers of A approaches the matrix T whose rows are each the fixed point t ;*
(3) *if u is any probability vector, then the sequence of vectors $uA, uA^2, uA^3, \cdots$ approaches the fixed point t.*

Proof. See [4]. □

NOTE : A^n approaches T means that each entry of A^n approaches the corresponding entry of T, and uA^n approaches t means that each component of uA^n approaches the corresponding component of t.

Definition 1.6. A stochastic matrix A is called a doubly-stochastic if not only the row sums but also the column sums are unity.

Let $SU_n(R^+) = \{A = (a_{ij}) \mid \sum_{k=1}^{n} a_{ik} = 1, \sum_{k=1}^{n} a_{kj} = 1\}$. Then $SU_n(R^+)$ is the set of all $n \times n$ doubly-stochastic matrices.

NOTE : If A and B are matrices in $SU_n(R^+)$, then AB is also in $SU_n(R^+)$, i.e. $SU_n(R^+)$ is closed with respect to multiplication.

2. Main Theorem

Now we prove the limit theorem of semi-positive doubly stochastic matrix that is not cyclic.

THEOREM 2.1. *If A is a semi-positive doubly stochastic matrix, then* $\lim_{m\to\infty} A^m = \frac{1}{n}J$ *where* $J = (a_{ij}), a_{ij} = 1$ *for all* i, j .

Proof. : Let $X = (x_1, x_2, \cdots, x_n)$ be a fixed probability vector of A. Then $XA = X$, $\sum_{i=1}^{n} x_i = 1$ and $x_i > 0$ for all i. From the equation $XA = X$, i.e. $(x_1, x_2, \cdots, x_n)A = (x_1, x_2, \cdots, x_n)$, we obtains a non-homogeneous system ;

$$(*) \quad \sum_{i=1}^{n} a_{ik}x_i = x_k \quad \text{for all} \quad k = 1, 2, \cdots, n \ .$$

Since $\sum_{i=1}^{n} x_i = 1$, $x_n = 1 - \sum_{i=1}^{n-1} x_i$. Thus we can write the equations as follows.

$$(**) \quad \sum_{i=1}^{n-1} a_{ik}x_i + a_{nk}(1 - \sum_{i=1}^{n-1} x_i) = x_k \ (k = 1, 2, \cdots, n) \ .$$

$$(***) \quad \sum_{i=1}^{n-1} a_{ik}x_i - a_{nk}\sum_{i=1}^{n-1} x_i - x_k = -a_{nk} \ (k = 1, 2, \cdots, n) \ .$$

If $k = n$, then $\sum_{i=1}^{n-1} a_{in}x_i - a_{nn}\sum_{i=1}^{n-1} x_i - (1 - \sum_{i=1}^{n-1} x_i) = -a_{nn}$.

That is, $\sum_{i=1}^{n-1}(a_{in} - a_{nn} + 1)x_i = 1 - a_{nn}$.

This equation is linearly dependent to the $(n-1)$ equations with $k = 1, 2, 3, \cdots, n-1$. Hence it is enough to solve $(n-1)$ equations. We use the Cramer's rule. Thus x_i is given by

$$\frac{\begin{vmatrix} a_{11}-a_{n1}-1 & a_{21}-a_{n1} & \cdots & -a_{n1} & \cdots & a_{n-1\,1}-a_{n1} \\ a_{12}-a_{n2} & a_{22}-a_{n2}-1 & \cdots & -a_{n2} & \cdots & a_{n-1\,2}-a_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1i}-a_{ni} & a_{2i}-a_{ni} & \cdots & -a_{ni} & \cdots & a_{n-1\,i}-a_{ni} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1\,n-1}-a_{n\,n-1} & a_{2\,n-1}-a_{n\,n-1} & \cdots & -a_{n\,n-1} & \cdots & a_{n-1\,n-1}-a_{n\,n-1}-1 \end{vmatrix}}{\begin{vmatrix} a_{11}-a_{n1}-1 & a_{21}-a_{n1} & \cdots & a_{i1}-a_{n1} & \cdots & a_{n-1\,1}-a_{n1} \\ a_{12}-a_{n2} & a_{22}-a_{n2}-1 & \cdots & a_{i2}-a_{n2} & \cdots & a_{n-1\,2}-a_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1i}-a_{ni} & a_{2i}-a_{ni} & \cdots & a_{ii}-a_{ni}-1 & \cdots & a_{n-1\,i}-a_{ni} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1\,n-1}-a_{n\,n-1} & a_{2\,n-1}-a_{n\,n-1} & \cdots & a_{i\,n-1}-a_{n\,n-1} & \cdots & a_{n-1\,n-1}-a_{n\,n-1}-1 \end{vmatrix}}$$

Replacing the $i-$th column in the denominator by the sum of all the columns and making the addition

$$\begin{aligned} & (a_{1i}-a_{ni})+(a_{2i}-a_{ni})+\cdots+(a_{ii}-a_{ni}-1)+(a_{i+1i}-a_{ni})+\cdots+(a_{n-1i}-a_{ni}) \\ = \; & (a_{1i}+a_{2i}+\cdots+a_{ii}+a_{i+1i}+\cdots+a_{n-1i})-(n-1)a_{ni}-1 \\ = \; & (1-a_{ni})-(n-1)a_{ni}-1=-na_{ni} \;\; (i=1,2,3,\cdots,n-1)\ . \end{aligned}$$

Then the denominator of x_i is equal to n times numerator of x_i. Hence $x_i=\frac{1}{n}$ for all $i=1,2,\cdots,n-1$. Thus $X=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$ is a fixed probability vector of A. Therefore,

$$\lim_{m\to\infty} A^m = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} = \frac{1}{n}J\ .$$

□

EXAMPLE 1. Let $A=\begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.5 & 0.1 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0.2 & 0.2 & 0.4 & 0.2 \end{pmatrix}$ and $X=(x_1,x_2,x_3,x_4)$ be a fixed probability vector of A.

Then

$$(0.1-0.2-1)x_1+(0.5-0.2)x_2+(0.2-0.2)x_3=-0.2$$
$$(0.3-0.2)x_1+(0.1-0.2-1)x_2+(0.4-0.2)x_3=-0.2$$
$$(0.2-0.4)x_1+(0.3-0.4)x_2+(0.1-0.4-1)x_3=-0.4\ .$$

Thus

$$\begin{cases} -1.1x_1+0.3x_2 & =-0.2 \\ 0.1x_1-1.1x_2+0.2x_3 & =-0.2 \\ -0.2x_1-0.1x_2-1.3x_3 & =-0.4\ . \end{cases}$$

Hence

$$\begin{cases} 11x_1-3x_2 & =2 \\ x_1-11x_2+2x_3 & =-2 \\ 2x_1+x_2+13x_3 & =4\ . \end{cases}$$

Thus $x_1=x_2=x_3=\frac{1}{4}=x_4$.

Therefore

$$\lim_{m\to\infty} A^m = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} .$$

In fact we can check this by computer calculation.

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