Kangweon-Kyungki Math. Jour. **11** (2003), No. 2, pp. 161–167

ON CHARACTERISTIC 0 AND WEAKLY ALMOST PERIODIC FLOWS

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Abstract. The purpose of this paper is to study and characterize the notions of characteristic 0 and weakly almost periodicity in flows. In particular, we give sufficient conditions for the weakly almost periodic flow to be almost periodic.

1. Introduction

The notions of distality, (uniformly) almost periodicity, pointwise almost periodicity, minimality, etc., have played a very crucial role in topological dynamics. Using these notions we study and characterize the notions of characteristic 0 and weakly almost periodicity in flows. In particular, we derive a number of similar dynamical consequences of characteristic 0 and weakly almost periodicity. Also we give sufficient conditions for the weakly almost periodic flow to be almost periodic and investigate some coalescent flows. Furthermore, for the weakly almost periodic flow (X, T) we find the groups I and $A(I)$ are isomorphic, where I is the minimal right ideal in $E(X)$, and $A(I)$ is the group of automorphisms of (X, T).

2. Preliminaries

Throughout this paper (X, T) will denote a flow, where the phase space X is assumed to be a compact Hausdorff space. The enveloping semigroup $E(X, T)$ (or simply $E(X)$) of the flow (X, T) is a kind of

Received July 18, 2003.
2000 Mathematics Subject Classification: 54H20.
Key words and phrases: characteristic 0, weakly almost periodic, almost periodic, distal, coalescent, proximally equicontinuous .
The present research has been conducted by the Research Grant of Kwangwoon University in 2002 .

compactification of the acting group and is itself a flow. It is easy to verify that $E(X \times X, T)$, as well as $E(E(X), T)$ are isomorphic with $E(X)$.

A flow is said to be *almost periodic* if the collection of maps defined by the action of the group is equicontinuous. A pair of points $(x, y), x, y \in X$ is *proximal* if $xp = yp$, for some $p \in E(X)$. A flow is *distal* if it has no nontrivial proximal pairs. The set of proximal pairs will be noted $P(X, T)$ or simply P. It is well known that the flow (X, T) is almost periodic iff $E(X)$ is a compact topological group and the elements of $E(X)$ are continuous maps. Also (X, T) is distal iff $E(X)$ is a group. Therefore it is obvious that if (X, T) is almost periodic, then it is distal. The flow (X, T) is said to be *proximally equicontinuous* if the proximal relation P is a closed equivalence relation on X, and if the quotient flow $(X/P, T)$ is equicontinuous. We denote the set of endomorphisms of (X, T) by $H(X)$, and the set of automorphisms of (X, T) by $A(X)$. If $A(X) = H(X)$, the flow (X, T) is said to be *coalescent*.

Definition 2.1. [10] A flow (X, T) (not necessarily minimal) is said to be *regular* if for each $x, y \in X$, there exists an endomorphism ϕ of (X, T) such that

(i) $(\phi(x), y) \in P(X, T)$, and

(ii) ϕ is an endomorphism on any minimal subset of (X, T).

Lemma 2.2. *Suppose $P(X, T)$ is an equivalence relation on X. Then we have the following [3].*

1. *$X = \cup_\alpha N_\alpha$, where the N_α are pairwise disjoint, $N_\alpha T \subset N_\alpha$, and each N_α contains precisely one minimal set M_α .*
2. *If $x \in N_\alpha$, then x is proximal to a point $y \in M_\alpha$.*
3. *If $P(X, T)$ is closed in $X \times X$, then the sets N_α are closed.*

Definition 2.3. Let (X, T) be a flow. We define the regionally proximal relation $Q(X, T) \subset X \times X$ by

$$Q(X, T) = \{(x, y) \mid \text{ there exist } x_i \to x,\ y_i \to y, \text{ and } t_i \in T \text{ such that } \lim x_i t_i = \lim y_i t_i \ \}.$$

3. Characteristic 0 and weakly almost periodicity

Definition 3.1. Let (X, T) be a flow. For each $x \in X$, we have the following definition [4] :

The first prolongation set $D(x)$ of x :
$D(x) = \{y \in X \mid x_i t_i \to y$ for some $x_i \to x$, and $t_i \in T\}$.

Definition 3.2. A point $x \in X$ is said to be of characteristic 0 under T if $D(x) = \overline{xT}$. The flow (X, T) is said to be of characteristic 0 if every point in X is of characteristic 0 under T [4].

Definition 3.3. A flow (X, T) is said to be 0-graphic if it is of characteristic 0, with T abelian such that any minimal subset of $(X \times X, T)$ is contained in a graph Γ_t for some $t \in T$ [7].

Definition 3.4. Let (X, T) be a flow. Given $f \in C(X)$ and $p \in E(X)$, set $({}_pf)(x) = f(xp)$, $\forall x \in X$. A function $f \in C(X)$ is said to be *weakly almost periodic* (w.a.p.) iff $({}_tf \mid t \in T)$ is relatively compact in the topology of pointwise convergence on $C(X)$.

A flow (X, T) is w.a.p. iff each $f \in C(X)$ is w.a.p. [6].

Lemma 3.5. *A flow (X, T) is w.a.p. iff each element of $E(X)$ is continuous [6].*

Remark 3.6. It follows from the above lemma that (X, T) is almost periodic, then (X, T) is w.a.p..

Lemma 3.7. *The following statements are pairwise equivalent [4].*
(1) The flow (X, T) is of characteristic 0.
(2) The relation $A = \{(x, y) \mid y \in \overline{xT}\}$ is a closed invariant equivalence relation (ICER) on X.

In [5], Ellis showed that if (X, T) is almost periodic, then A is an ICER on X. Hence, by the above lemma, we have the following corollary.

Corollary 3.8. *Let (X, T) be almost periodic. Then (X, T) is of characteristic 0.*

Remark 3.9. Note that (X, T) is almost periodic iff the orbit closure relation on $(X \times X, T)$ is an ICER on $X \times X$ [5].

Remark 3.10. Note that products and facters of w.a.p. flows are again w.a.p. while the property of characteristic 0 is not preserved under the product operation [4; Example 5.4]. Also note that if $(X \times X, T)$ is of characteristic 0, then (X, T) is of characteristic 0 [4; Theorem 2.7].

LEMMA 3.11. *Let (X, T) be w.a.p. and let I be a minimal right ideal in $E(X)$. Then we have the following [6].*

1. *(I, T) is almost periodic.*
2. *I has a unique idempotent u.*
3. *I is a compact topological group with identity u.*
4. *I is also a minimal left ideal.*
5. *I is the only minimal right ideal in $E(X)$.*
6. *Let $p \in E(X)$, then $pu = up$.*

LEMMA 3.12. *Let (X, T) be w.a.p.. Then $P(X, T)$ is an ICER on X [9].*

THEOREM 3.13. *Let (X, T) be w.a.p.. Then there exists a partition of X consisting of compact Hausdorff invariant subsets and each this partition elements contains exactly one minimal set.*

Proof. This follows from Lemma 3.12 and Lemma 2.2. □

THEOREM 3.14. *Let (X, T) be of characteristic 0. Then there exists a partition of X consisting of compact minimal subsets.*

Proof. Let $x \in X$ and let $y \in \overline{xT}$. Then $y \in D(x)$. From the definition of $D(x)$ it follows that $x \in D(y) = \overline{yT}$. Hence we have $\overline{xT}$ is a compact minimal subset of X for all $x \in X$. Since minimal subsets are disjoint or equal, it follows that $\{\overline{xT} \mid x \in X\}$ forms the desired partition. □

REMARK 3.15. It is well known that x is an almost periodic point iff $\overline{xT}$ is a compact minimal subset of X [5; Proposition 2.5]. Hence note that if (X, T) is of characteristic 0, then it is pointwise almost periodic. Also that if (X, T) is distal, then it is pointwise almost periodic [5; Corollary 5.5].

THEOREM 3.16. *If (X, T) is w.a.p., then it is proximally equicontinuous.*

Proof. As we observed in Lemma 3.12 $P(X, T)$ is the smallest ICER on X which contains $P(X, T)$. Hence $(X/P, T)$ is distal [5; Theorem 5.23] and $E(X/P)$ is a group. Since a factor of a w.a.p. flow is w.a.p., we have the flow $(X/P, T)$ is also w.a.p. and each element of $E(X/P)$ is continuous. This shows that $(X/P, T)$ is almost periodic. □

REMARK 3.17. It is well known that $(x, y) \in P(X, T)$ iff $\overline{(x, y)T} \bigcap \triangle \neq \emptyset$ [5; Remark 5.11]. Also it follows from Definition 3.1 that $(x, y) \in Q(X, T)$ iff $D(x, y) \bigcap \triangle \neq \emptyset$.

THEOREM 3.18. *If $(X \times X, T)$ is of characteristic 0, then it is proximally equicontinuous.*

Proof. It follows from Definition 3.2 and Remark 3.17 that $P(X,T) = Q(X,T)$. Then $P(X,T)$ is the smallest ICER on X which contains $Q(X,T)$. Hence $(X/P,T)$ is almost periodic [5; Theorem 4.20]. □

THEOREM 3.19. *Let (X,T) be w.a.p.. If (X,T) is pointwise almost periodic, then it is almost periodic.*

Proof. Let I be the only minimal right ideal in $E(X)$ and let u be a unique idempotent in I. Since (X,T) is pointwise almost periodic, we have $xu = x$ for all $x \in X$. Now let $p \in E(X)$. Then $xp = (xu)p = x(up)$ for all $x \in X$. This means that $p = up \in I$. Thus $I = E(X)$. Since (X,T) is w.a.p., it follows that $E(X)$ is a group of continuous maps of X into X. Hence (X,T) is almost periodic. □

Auslander's result [2; Theorem 6 of chapter 4] is a corollary to the above theorem :

COROLLARY 3.20. *If (X,T) is a w.a.p. minimal flow, then it is almost periodic.*

Proof. Note that if (X,T) is a minimal flow, then it is pointwise almost periodic. □

The following corollaries are immediate.

COROLLARY 3.21. *If (X,T) is of characteristic 0 and w.a.p., then it is almost periodic.*

Proof. Note that if (X,T) is of characteristic 0, then $\overline{xT}$ is minimal for all $x \in X$. □

COROLLARY 3.22. *If (X,T) is w.a.p. distal, then it is almost periodic.*

Proof. The proof is straight forward and is thus omitted. □

THEOREM 3.23. *Let (X,T) be a w.a.p. flow with T abelian. Then every orbit closure is regular.*

Proof. Since (X,T) is w.a.p., we have $P(X,T)$ is an equivalence relation on X and there is an minimal right ideal of $E(X)$ all of whose elements are continuous. Since T is abelian, by [10; Theorem 19] it follows that every orbit closure is regular. □

Corollary 3.24. *Let (X, T) be a w.a.p. minimal flow with T abelian. Then it is regular minimal, and $(E(X), T)$ is isomorphic with (X, T).*

Proof. This follows from Corollary 3.20, Theorem 3.23 and [5; Remark 4.6]. □

In [8], Song studied the flows of characteristic 0 and proved the following theorem :

Theorem 3.25. *The following statements are valid [8].*

1. *The homomorphic image of a diatal flow of characteristic 0 is a distal flow of characteristic 0.*
2. *Let $\varphi : (X, T) \longrightarrow (Y, T)$ be an epimorphism of flows of characteristic 0. Then (X, T) is 0-graphic if and only if (Y, T) is 0-graphic.*
3. *A 0-graphic flow is regular.*
4. *If (X, T) is a 0-graphic flow, then every orbit closure is regular minimal.*

Corollary 3.26. *Let (X, T) be a 0-graphic minimal flow. Then it is regular minimal, and (X, T) is isomorphic with (I, T), where I is a minimal right ideal in $E(X)$.*

Note that if (X, T) is of characteristic 0, then it is pointwise almost periodic.

Theorem 3.27. *Let (X, T) be w.a.p. and pointwise almost periodic. Then every orbit closure is coalescent.*

Proof. Since (X, T) is pointwise almost periodic, it follows that $(\overline{xT}, T)$ is minimal for all $x \in X$. But since (X, T) is w.a.p., by [1; Theorem 7], we have $(\overline{xT}, T)$ is coalescent. □

The following corollaries are immediate.

Corollary 3.28. *Weakly almost periodic minimal sets are coalescent.*

Corollary 3.29. *Almost periodic minimal sets are coalescent.*

Theorem 3.30. *Let (X, T) be w.a.p. and let I be the only minimal right ideal in $E(X)$. Then the groups I and $A(I)$ are isomorphic.*

Proof. Let $\varphi \in H(I)$. Then there exists a $p \in I$ such that $\varphi = L_p$ [1; Theorem 3]. Since left cancellation holds in a minimal right ideal, we have $L_p \in A(I)$. Also it is easy to see that $L_p = L_q$ if and only if $p = qu$ for some $u^2 = u \in I$. Since (X, T) is w.a.p., it follows that I has a unique idempotent u and I is a group with identity u. Then the map $\phi : p \mapsto L_p$ of I onto $A(I)$ is bijective and $\phi(pq) = L_{pq} = L_p L_q = \phi(p)\phi(q)$. This means that I and $A(I)$ are isomorphic. □

Corollary 3.31. *Let (X, T) be almost periodic. Then the groups $E(X)$ and $A(E(X))$ are isomorphic.*

Proof. Note that if (X, T) is almost periodic, then it is distal. Hence $E(X)$ is a minimal right ideal and a group. Then by the above theorem, we have $E(X)$ is isomorphic with $A(E(X))$. □

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