G-FUZZY EQUIVALENCE RELATIONS
GENERATED BY FUZZY RELATIONS

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Abstract. We find a G-fuzzy equivalence relation generated by
the union of two G-fuzzy equivalence relations in a set, find a G-
fuzzy equivalence relation generated by a fuzzy relation in a set, and
find sufficient conditions for the composition $\mu \circ \nu$ of two G-fuzzy
equivalence relations $\mu$ and $\nu$ to be a G-fuzzy equivalence relation
generated by $\mu \cup \nu$.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([6]).
Subsequently, Goguen ([1]) and Sanchez ([5]) studied fuzzy relations in
various contexts. In [4] Nemitz discussed fuzzy equivalence relations,
fuzzy functions as fuzzy relations, and fuzzy partitions. Murali ([3])
developed some properties of fuzzy equivalence relations and certain
lattice theoretic properties of fuzzy equivalence relations. Gupta et al.
([2]) proposed a generalized definition of a fuzzy equivalence relation
on a set, which we call G-fuzzy equivalence relation in this paper, and
developed some properties of that relation. The present work has been
started as a continuation of these studies.

In section 2 we develop some basic properties of fuzzy relations,
find a G-fuzzy equivalence relation generated by the union of two G-
fuzzy equivalence relations in a set, find a G-fuzzy equivalence relation
generated by a fuzzy relation in a set, and show that if $\mu$ and $\nu$ are G-
fuzzy equivalence relations in a set such that $\mu \circ \nu = \nu \circ \mu$, $\inf_{t \in X} \mu(t, t) \geq
\nu(x, y)$, and $\inf_{t \in X} \nu(t, t) \geq \mu(x, y)$ for all $x \neq y \in X$, then $\mu \circ \nu$ is a
G-fuzzy equivalence relation generated by $\mu \cup \nu$.

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2. Fuzzy equivalence relation

Definition 2.1. A function $B$ from a set $X$ to the closed unit interval $[0, 1]$ in $\mathbb{R}$ is called a fuzzy set in $X$. For every $x \in B$, $B(x)$ is called a membership grade of $x$ in $B$.

The standard definition of a fuzzy reflexive relation $\mu$ in a set $X$ demands $\mu(x, x) = 1$. Gupta et al. ([2]) weakened this definition as follows.

Definition 2.2. A fuzzy relation $\mu$ in a set $X$ is a fuzzy subset of $X \times X$. $\mu$ is $G$-reflexive in $X$ if $\mu(x, x) > 0$ and $\mu(x, y) \leq \inf_{t \in X} \mu(t, t)$ for all $x \neq y$ in $X$. $\mu$ is symmetric in $X$ if $\mu(x, y) = \mu(y, x)$ for all $x, y$ in $X$. The composition $\lambda \circ \mu$ of two fuzzy relations $\lambda, \mu$ in $X$ is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x, y) = \sup_{z \in X} \min(\lambda(x, z), \mu(z, y)).$$

A fuzzy relation $\mu$ in $X$ is transitive in $X$ if $\mu \circ \mu \subseteq \mu$. A fuzzy relation $\mu$ in $X$ is called $G$-fuzzy equivalence relation if $\mu$ is $G$-reflexive, symmetric, and transitive.

Proposition 2.3. Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set $X$. Then $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$.

Proof. Clearly $\circ$ is a binary operation. It is well known that $\circ$ is associative (see Proposition 2.3 of [3]). Let $\theta$ be a fuzzy relation such that $\theta(x, x) = 1$ and $\theta(x, y) = 0$ if $x \neq y$. Then $(\mu \circ \theta)(x, y) = \sup_{z \in X} \min(\mu(x, z), \theta(z, y)) = \mu(x, y)$. Similarly we may show $(\theta \circ \mu)(x, y) = \mu(x, y)$. Hence $\mathcal{F}_X$ is a monoid. \hfill $\square$

It is easy to see that a $G$-fuzzy equivalence relation is an idempotent element of $\mathcal{F}_X$.

Definition 2.4. Let $\mu$ be a fuzzy relation in a set $X$. $\mu^{-1}$ is defined as a fuzzy relation in $X$ by $\mu^{-1}(x, y) = \mu(y, x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations $\mu$ and $\nu$. 
Proposition 2.5. Let \( \mathcal{F}_X \) be a monoid of all fuzzy relations in \( X \) and let \( \phi : \mathcal{F}_X \to \mathcal{F}_X \) be a map defined by \( \phi(\mu) = \mu^{-1} \). Then \( \phi \) is an antiautomorphism and \( \phi(\mu^{-1}) = (\phi(\mu))^{-1} = \mu \).

Proof. Since \((\mu^{-1})^{-1}(x,y) = \mu^{-1}(y,x) = \mu(x,y)\) for all \( x,y \in X \), \( \phi(\mu^{-1}) = (\mu^{-1})^{-1} = \mu = (\phi(\mu))^{-1} \). Since \((\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1} \), \( \phi(\mu \circ \nu) = (\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1} = \phi(\nu) \circ \phi(\mu) \). □

Proposition 2.6. Let \( \mu \) and each \( \nu_i \) be fuzzy relations in a set \( X \) for all \( i \in I \). Then \( \mu \circ (\bigcup_{i \in I} \nu_i) = \bigcup_{i \in I} (\mu \circ \nu_i) \), \( (\bigcup_{i \in I} \nu_i) \circ \mu = \bigcup_{i \in I} (\nu_i \circ \mu) \), \( \mu \circ (\bigcap_{i \in I} \nu_i) \subseteq \bigcap_{i \in I} (\mu \circ \nu_i) \), and \( (\bigcap_{i \in I} \nu_i) \circ \mu \subseteq \bigcap_{i \in I} (\nu_i \circ \mu) \).

Proof.

\[
[\mu \circ (\bigcup_{i \in I} \nu_i)](x,y) = \sup_{z \in X} \min[\mu(x,z), (\bigcup_{i \in I} \nu_i)(z,y)] \\
= \sup_{z \in X} \min[\mu(x,z), \sup_{i \in I} \nu_i(z,y)] \\
= \sup_{i \in I} \sup_{z \in X} \min[\mu(x,z), \nu_i(z,y)] \\
= \sup_{i \in I} \sup_{z \in X} \min[\mu(x,z), \nu_i(z,y)] \\
= (\bigcup_{i \in I} \mu \circ \nu_i)(x,y).
\]

Similarly we may prove the remaining things. □

Proposition 2.7. Let \( \mu \) and \( \nu \) be \( G \)-fuzzy equivalence relations in a set \( X \). Then \( \mu \cap \nu \) is a \( G \)-fuzzy equivalence relation.

Proof. \((\mu \cap \nu)(x,x) = \min(\mu(x,x), \nu(x,x)) > 0\).

\[
\inf_{t \in X} (\mu \cap \nu)(t,t) = \inf_{t \in X} \min(\mu(t,t), \nu(t,t)) \\
= \min (\inf_{t \in X} \mu(t,t), \inf_{t \in X} \nu(t,t)) \\
\geq \min (\mu(x,y), \nu(x,y)) = (\mu \cap \nu)(x,y)
\]

for all \( x \neq y \) in \( X \). That is, \( \mu \cap \nu \) is \( G \)-reflexive. \((\mu \cap \nu)(x,y) = \min(\mu(x,y), \nu(x,y)) = \min(\mu(y,x), \nu(y,x)) = (\mu \cap \nu)(y,x)\). By Proposition 2.6, \([\mu \cap \nu] \circ (\mu \cap \nu) \subseteq [\mu \circ (\mu \cap \nu)] \cap [\nu \circ (\mu \cap \nu)] \subseteq [(\mu \circ \mu) \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap (\nu \circ \nu)] \subseteq [\mu \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap \nu] \subseteq \mu \cap \nu \). □
It is easy to see that even though $\mu$ and $\nu$ are G-fuzzy equivalence relations, $\mu \cup \nu$ is not necessarily a G-fuzzy equivalence relation. We find a G-fuzzy equivalence relation generated by $\mu \cup \nu$ in the following theorem.

**Theorem 2.8.** Let $\mu$ and $\nu$ be G-fuzzy equivalence relations in a set $X$. The G-fuzzy equivalence relation generated by $\mu \cup \nu$ is $\bigcup_{n=1}^{\infty}(\mu \cup \nu)^n = (\mu \cup \nu) \cup [(\mu \cup \nu) \circ (\mu \cup \nu)] \cup \ldots$.

**Proof.** Clearly $(\mu \cup \nu)(x, x) > 0$.

$$\inf_{t \in X} (\mu \cup \nu)(t, t) = \inf_{t \in X} \max(\mu(t, t), \nu(t, t))$$

$$= \max (\inf_{t \in X} \mu(t, t), \inf_{t \in X} \nu(t, t))$$

$$\geq \max (\mu(x, y), \nu(x, y)) = (\mu \cup \nu)(x, y)$$

for all $x \neq y$ in $X$. That is, $\mu \cup \nu$ is G-reflexive. Similarly $(\mu \cup \nu)^n$ is G-reflexive for $n = 3, 4, \ldots$. Clearly $(\mu \cup \nu)(x, x) = \sup_{z \in X} \min[(\mu \cup \nu)(x, z), (\mu \cup \nu)(z, x)] \geq \min[(\mu \cup \nu)(x, x), (\mu \cup \nu)(x, x)] > 0$. Hence $\inf_{t \in X} (\mu \cup \nu)^n(t, t) = \inf_{t \in X} \sup_{z \in X} \min[(\mu \cup \nu)^n(t, z), (\mu \cup \nu)(z, t)] \geq \inf_{t \in X} \min[(\mu \cup \nu)(t, t), (\mu \cup \nu)(t, t)] = \inf_{t \in X} (\mu \cup \nu)(t, t) \geq \sup_{z \in X} \min[(\mu \cup \nu)(x, z), (\mu \cup \nu)(z, y)] = ((\mu \cup \nu) \circ (\mu \cup \nu))(x, y).$ That is, $(\mu \cup \nu) \circ (\mu \cup \nu)$ is G-reflexive. Similarly $(\mu \cup \nu)^n$ is G-reflexive for $n = 3, 4, \ldots$. Clearly $(\mu \cup \nu)^n(x, x) > 0$.

Thus $\bigcup_{n=1}^{\infty}(\mu \cup \nu)^n$ is G-reflexive. Clearly $\mu \cup \nu$ is symmetric. Clearly $(\mu \cup \nu) \circ (\mu \cup \nu)(x, y) = \sup_{z \in X} \min[(\mu \cup \nu)(x, z), (\mu \cup \nu)(z, y)] = [(\mu \cup \nu) \circ (\mu \cup \nu)](y, x)$. That is, $(\mu \cup \nu) \circ (\mu \cup \nu)$ is symmetric. Similarly we may show $(\mu \cup \nu)^n$ is symmetric for $n = 3, 4, \ldots$. Let $\mu^n = \bigcup_{n=1}^{\infty}(\mu \cup \nu)^n$ and let $\mu_1 = \mu \cup \nu$. By Proposition 2.6, $\mu \ast \mu = \mu^2 \circ (\mu \cup \nu) \cup \ldots \cup ([\mu \ast \mu_1] \cup [\mu \ast (\mu_1 \cup \mu_1)] \cup \ldots) = ([\mu \ast \mu_1] \cup [\mu \ast (\mu_1 \cup \mu_1)] \cup \ldots) \cup \ldots$
\[(\mu_1 \circ (\mu_1 \circ \mu_1)) \cup (\mu_1 \circ (\mu_1 \circ \mu_1)) \cup \ldots \] \cup \cdots = [(\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cup \ldots] \cup [(\mu_1 \circ \mu_1 \circ \mu_1) \cup \ldots] \cup \cdots \subseteq \mu^* . \] That is, \(\mu^* = \cup_{n=1}^{\infty} (\mu \cup \nu)^n\) is transitive. Hence \(\cup_{n=1}^{\infty} (\mu \cup \nu)^n\) is a G-fuzzy equivalence relation. Let \(\lambda\) be a G-fuzzy equivalence relations in a set \(X\) containing \(\mu \cup \nu\). Then \(\cup_{n=1}^{\infty} (\mu \cup \nu)^n \subseteq \cup_{n=1}^{\infty} \lambda^n = \lambda \cup (\lambda \circ \lambda) \cup (\lambda \circ \lambda \circ \lambda) \cup \ldots \subseteq \lambda \cup \lambda \cup \ldots = \lambda\). That is, \(\cup_{n=1}^{\infty} (\mu \cup \nu)^n\) is contained in every G-fuzzy equivalence relation in \(X\) containing \(\mu \cup \nu\). Thus \(\cup_{n=1}^{\infty} (\mu \cup \nu)^n\) is a G-fuzzy equivalence relation generated by \(\mu \cup \nu\).

**Theorem 2.9.** Let \(\mu\) be a fuzzy relation in a set \(X\). Then G-fuzzy equivalence relation in \(X\) generated by \(\mu\) is \(\mu^* = \cup_{n=1}^{\infty} \mu_1^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cup \ldots, \) where \(\mu_1 = \mu \cup \mu^{-1} \cup \theta\) and \(\theta\) is a fuzzy relation in \(X\) such that \(\theta(x, x) > 0, \theta = \theta^{-1}, \theta(x, y) \leq \mu(x, y), \) and max[\(\mu(x, y), \theta(x, y)\)] \(\leq \inf_{t \in X} \theta(t, t)\) for all \(x \neq y\) in \(X\).

**Proof.** \(\inf_{t \in X} (\mu \cup \mu^{-1} \cup \theta)(t, t) = \inf_{t \in X} \max[\mu(t, t), \mu^{-1}(t, t), \theta(t, t)] \geq \inf_{t \in X} \theta(t, t) \geq \max[\mu(\mu(x, y), \mu^{-1}(x, y), \theta(x, y)] = (\mu \cup \mu^{-1} \cup \theta)(x, y).

Thus \(\mu_1 = \mu \cup \mu^{-1} \cup \theta\) is G-reflexive. By the same way as shown in Theorem 2.8, we may show \(\mu^* = \cup_{n=1}^{\infty} \mu_1^n\) is G-reflexive.

\[\mu_1(x, y) = (\mu \cup \mu^{-1} \cup \theta)(x, y) = \max[\mu(x, y), \mu^{-1}(x, y), \theta^{-1}(x, y)] = \max[\mu^{-1}(y, x), \mu(y, x), \theta(y, x)] = (\mu \cup \mu^{-1} \cup \theta)(y, x) = \mu_1(y, x).

Thus \(\mu_1\) is a symmetric. By the same way as shown in Theorem 2.8, we may show \(\mu^* = \cup_{n=1}^{\infty} \mu_1^n\) is symmetric and transitive. Hence \(\mu^*\) is a G-fuzzy equivalence relation containing \(\mu\). Let \(\nu\) be a G-fuzzy equivalence relation containing \(\mu\). Then \(\mu(x, y) \leq \nu(x, y), \mu^{-1}(x, y) = \mu(y, x) \leq \nu(y, x) = \nu(x, y), \) and \(\theta(x, y) \leq \mu(x, y) \leq \nu(x, y)\). Thus \(\mu_1 = (\mu \cup \mu^{-1} \cup \theta) \subseteq \nu, (\mu_1 \circ \mu_1)(x, y) = \sup_{z \in X} \min[\mu_1(x, z), \mu_1(z, y)] \leq \sup_{z \in X} \min[\nu(x, z), \nu(z, y)] = (\nu \circ \nu)(x, y)\). Since \(\nu\) is transitive, \(\mu_1 \circ \mu_1 \subseteq \nu \circ \nu \subseteq \nu\). Similarly we may show \(\mu_1^n \subseteq \nu\) for \(n = 3, \ldots\). Thus \(\mu^* = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu\).
Theorem 2.10. Let \( \mu \) and \( \nu \) be G-fuzzy equivalence relations in a set \( X \) such that \( \inf_{t \in X} \mu(t, t) \geq \nu(x, y) \) and \( \inf_{t \in X} \nu(t, t) \geq \mu(x, y) \) for all \( x \neq y \in X \). If \( \mu \circ \nu = \nu \circ \mu \), then \( \mu \circ \nu \) is a G-fuzzy equivalence relation in \( X \) generated by \( \mu \cup \nu \).

Proof.

\[
(\mu \circ \nu)(x, x) = \sup_{z \in X} \min[\mu(x, z), \nu(z, x)]
\geq \min(\mu(x, x), \nu(x, x)) > 0.
\]

Since \( \inf_{t \in X} \mu(t, t) \geq \nu(x, y) \) and \( \inf_{t \in X} \nu(t, t) \geq \mu(x, y) \) for all \( x \neq y \in X \),

\[
\inf_{t \in X} (\mu \circ \nu)(t, t) = \inf_{t \in X} \sup_{z \in X} \min[\mu(t, z), \nu(z, t)]
\geq \inf_{t \in X} \min[\mu(t, t), \nu(t, t)] \geq \min[\mu(x, z), \nu(z, y)]
\]

for all \( z \in X \). Thus \( \inf_{t \in X} (\mu \circ \nu)(t, t) \geq \sup_{z \in X} \min[\mu(x, z), \nu(z, y)] = (\mu \circ \nu)(x, y) \). That is, \( \mu \circ \nu \) is G-reflexive. Since \( \mu \) and \( \nu \) are symmetric, \( (\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1} = \nu \circ \mu = \mu \circ \nu \). Thus \( \mu \circ \nu \) is symmetric. Since \( \mu \) and \( \nu \) are transitive and the operation \( \circ \) is associative, \( (\mu \circ \nu) \circ (\mu \circ \nu) = \mu \circ (\nu \circ \mu) \circ \nu = \mu \circ (\mu \circ \nu) \circ \nu = (\mu \circ \mu) \circ (\nu \circ \nu) \subseteq \mu \circ \nu \). Hence \( \mu \circ \nu \) is a G-fuzzy equivalence relation. Since \( \nu(y, y) \geq \mu(x, y) \), \( (\mu \circ \nu)(x, y) = \sup_{z \in X} \min[\mu(x, z), \nu(z, y)] \geq \min[\mu(x, y), \nu(y, y)] = \mu(x, y) \). Since \( \mu(x, x) \geq \nu(x, y) \), \( (\mu \circ \nu)(x, y) = \sup_{z \in X} \min[\mu(x, z), \nu(z, y)] \geq \min(\mu(x, x), \nu(x, y)) = \nu(x, y) \). Thus \( (\mu \circ \nu)(x, y) \geq \max(\mu(x, y), \nu(x, y)) = (\mu \cup \nu)(x, y) \). That is, \( \mu \cup \nu \subseteq \mu \circ \nu \).

Let \( \lambda \) be a G-fuzzy equivalence relation in \( X \) containing \( \mu \cup \nu \). Since \( \lambda \) is transitive, \( \mu \circ \nu \subseteq (\mu \cup \nu) \circ (\mu \cup \nu) \subseteq \lambda \circ \lambda \subseteq \lambda \). Thus \( \mu \circ \nu \) is a G-fuzzy equivalence relation generated by \( \mu \cup \nu \). \( \square \)

References


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