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PUSHCHINO DYNAMICS OF INTERNAL LAYER

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Abstract. The existence of solutions and the occurence of a Hopf bifurcation for the free boundary problem with Pushchino dynamics was shown in [3]. In this paper we shall show a Hopf bifurcation occurs for the free boundary which is given by (1)

1. Introduction

In [3], they deal with the free boundary problem with Pushchino dynamics. They showed the existence of solutions and the occurence of a Hopf bifurcation. In this paper we shall show a Hopf bifurcation occurs for the free boundary which is given by (1)(see in [2])

$$(1)\quad \begin{cases} v_t = v_{xx} - (c_1 + b)v + c_1 H(x - s(t)) + \kappa, (x,t) \in \Omega^- \cup \Omega^+, \\ v_x(0,t) = 0 = v_x(1,t), \;\; t > 0, \\ v(x,0) = v_0(x), \;\; 0 \le x \le 1, \\ \tau \frac{ds}{dt} = C(v(s(t),t)), \;\; t > 0, \\ s(0) = s_0, \;\; 0 < s_0 < 1, \end{cases}$$

where $v(x,t)$ and $v_x(x,t)$ are assumed continuous in $\Omega = (0,1) \times (0,\infty)$. Here, $H(\cdot)$ is the Heaviside function, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < 1\}$. The velocity of the interface, $C(v)$, in (1), which specifies the evolution of the interface $s(t)$,

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is determined from the first equation in (1) using asymptotic techniques (see in [1]). The function $C(v)$ can be calculated explicitly as

$$C(v) = \frac{2(c_1+c_2)v - 2\kappa - (c_1 - 2a)}{\sqrt{(\frac{c_1-a+\kappa}{c_1+c_2} - v)(v + \frac{a-\kappa}{c_1+c_2})}}$$

where $0 < a < 1/2$ and c_1, c_2 and κ are positive constants. We assume that the bistability, $-c_1 < b < \frac{c_1(c_2+a)}{c_1+a} + \kappa$.

2. The preliminary results

We recall the few things from [4]:

Let $G(x, y)$ be Green's function of the operator $A := -\frac{d^2}{dx^2} + (c_1 + b)$ and the domain of the operator A, $D(A) = \{v \in H^{2,2}(0,1) : v_x(0) = v_x(1) = 0\}$. Define a function

$$g(x, s) := \int_0^1 G(x, y)(c_1\, H(y - s) + \kappa\,)dy$$

and $\gamma(s) = g(s, s)$. We obtain the regular problem of (1) by using the transformation $u(x, t) = v(x, t) - g(x, s(t))$:

$$\begin{cases} \frac{d}{dt}(u, s) + \tilde{A}(u, s) = \dfrac{1}{\tau}\, f(u, s) \\ (u, s)(0) = (u_0, s_0). \end{cases} \tag{2}$$

The operator $\tilde{A}$ is a 2×2 matrix whose the entry of the first row and column is the operator A and the rest terms are all zero. The nonlinear term $f(u, s)$ is represented by

$$f(u, s) = \begin{pmatrix} c_1\, C(u(s) + \gamma(s))\, G(x, s) \\ C(u(s) + \gamma(s)) \end{pmatrix}.$$

We shall show the stationary solution of (1) (or (2)) exists and the Hopf bifurcation occurs in the next section.

3. The Hopf bifurcation

3.1 The stationary solutions

Let $u(x,t) = u^*(x)$ and $s(t) = s^*$ and the time derivatives in (2) equal to zero we obtain the stationary problem:

$$(3) \qquad \begin{cases} Au^* &= \frac{c_1}{\tau}\, C(u^*(s^*) + \gamma(s^*))\, G(\cdot, s^*) \\ 0 &= \frac{1}{\tau}\, C(u^*(s^*) + \gamma(s^*)) \end{cases} .$$

For nonzero τ we obtain the following theorem:

Theorem 1. *Assume that*

$$\frac{(c_1 - 2a)(c_1 + b) - 2c_1(c_1 + c_2)}{2(c_2 - b)} < \kappa < \frac{(c_1 - 2a)(c_1 + b)}{2(c_2 - b)}$$

then (2) has a unique stationary solution $(u^*(x), s^*) = (0, s^*)$ *for all* $0 < \tau < \infty$. *The linearization of* f *at* $(0, s^*)$ *is*

$$Df(0, s^*)(\hat{u}, \hat{s}) = \frac{4(c_1+c_2)^2}{c_1}\, (\hat{u}(s^*) + \gamma'(s^*)\, \hat{s})\big(G(s^*, s^*), 1\big).$$

The pair $(0, s^*)$ *corresponds to a unique steady state* (v^*, s^*) *of (1) for* $\tau \neq 0$ *with* $v^*(x) = g(x, s^*)$.

Proof. From the (3), we easily see that $u^* = 0$ and s^* is a solution of the equation $\gamma(s) = \frac{c_1 - 2a + 2\kappa}{2(c_1 + c_2)}$. Now, we define

$$\Gamma(s) := \gamma(s) - \frac{c_1 - 2a + 2\kappa}{2(c_1 + c_2)}$$

where $\gamma(s) = \frac{c_1}{(c_1+b)\ \sinh \sqrt{c_1+b}}\ \cosh(\sqrt{c_1 + b}\ s)\ \sinh(\sqrt{c_1 + b}\,(1 - s)) + \frac{\kappa}{c_1+b}$. Since $\Gamma'(s) < 0$ for $s \in (0, 1)$, we need the following conditions $\Gamma(0) > 0$ and $\Gamma(1) < 0$ and thus we obtain the above condition. The formula for $Df(0, s^*)$ follows from the differentiation and the relation $C'(\frac{c_1 - 2a + 2\kappa}{2(c_1+c_2)}) = \frac{4(c_1+c_2)^2}{c_1}$. Using Theorem 2.4 in [4], we obtain the corresponding steady state (v^*, s^*) for (1). ☐

3.2 A Hopf bifurcation

We now show that a Hopf bifurcation occurs as the new parameter $\mu, \mu = \frac{1}{\tau}\frac{c_1}{4(c_1+c_2)^2}$ varies. The linearized eigenvalue problem of (2) is given by

$$(-\tilde{A} + \mu Df(0,s^*))(u,s) = \lambda(u,s)$$

which is equivalent to

$$Au + \lambda u = \mu\, c_1\,(u(s^*) + \gamma'(s^*)s)G(x,s^*) \tag{4}$$

$$\lambda s = \mu(u(s^*) + \gamma'(s^*)s) \tag{5}$$

We have the following lemma:

LEMMA 2. *For $\mu^* \in \mathbb{R}\setminus\{0\}$, there is a C^1-curve $\mu \to (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$ where ϕ^* is an eigenfunction of $-\tilde{A} + \mu^* Df(0,s^*)$ with eigenvalue $i\beta$.*

Proof. Let $\phi^* = (\psi_0, s_0) \in D(A)\times\mathbb{R}$. First, we see that $s_0 \neq 0$, for otherwise, by (4), $(A+i\beta)\psi_0 = i\beta\, c_1\, G(\cdot,s^*)s_0 = 0$, which is not possible because A is symmetric. So without loss of generality, let $s_0 = 1$. Then by (4) $E(\psi_0, i\beta, \mu^*) = 0$, where

$$E : D(A)_{\mathbb{C}} \times \mathbb{C} \times \mathbb{R} \longrightarrow X_{\mathbb{C}} \times \mathbb{C}\,,$$

$$E(u,\lambda,\mu) = \begin{pmatrix} (A+\lambda)u - \mu\, c_1\,(u(s^*) + \gamma'(s^*))\,G(\cdot,s^*) \\ \lambda - \mu\cdot(u(s^*) + \gamma'(s^*)) \end{pmatrix}.$$

The equation $E(u,\lambda,\mu) = 0$ is equivalent that λ is an eigenvalue of $-\widetilde{A} + \mu Df(0,s^*)$ with eigenfunction $(u,1)$. We want to apply the implicit function theorem to E, and therefore have to check that E is in C^1 and that

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu_0) : D(A)_{\mathbb{C}} \times \mathbb{C} \to L^2(0,1)\times\mathbb{C}) \tag{6}$$

is an isomorphism. Now it is easy to see that E is in C^1. The mapping

$$\begin{aligned} &D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat{u},\hat{\lambda}) \\ &\quad = \begin{pmatrix} (A+i\beta)\hat{u} - \mu^* c_1 \hat{u}(s^*)\cdot G(\cdot,s^*) + \hat{\lambda}\psi_0 \\ -\mu^*\hat{u}(s^*) + \hat{\lambda} \end{pmatrix} \end{aligned}$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \longmapsto \Big((A + i\beta)\hat{u}, \hat{\lambda}\Big)$$

which is invertible. As a consequence, $D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)$ is a Fredholm operator of index 0. Thus to verify (6), it suffices to show that the system

$$\begin{cases} (A + i\beta)\hat{u} + \hat{\lambda}\psi_0 = \mu^* \hat{u}(s^*) c_1 G(\cdot, s^*) \\ \hat{\lambda} = \mu^* \hat{u}(s^*) \end{cases} \tag{7}$$

necessarily implies that $\hat{u} = 0$, $\hat{\lambda} = 0$. Thus let $(\hat{u}, \hat{\lambda})$ be a solution of (7), and define $\psi_1 := \psi_0 - c_1 G(\cdot, s^*)$. Then

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0\,. \tag{8}$$

Also, ψ_1 is a solution to the equation

$$(A + i\beta)\psi_1 = -c_1 \delta_{s^*} \tag{9}$$

$$i\beta = \mu^* \cdot \Big(\gamma^{'}(s^*) + \psi_1(s^*) + c_1 G(s^*, s^*)\Big) \tag{10}$$

where δ_s is the delta-distribution centered at s. From the equation (9), we have

$$\mathrm{Im}\,(\psi_1(s^*)) = \beta \int_0^1 |\psi_1|^2\,.$$

So, we have that

$$\mu^* \int_0^1 |\psi_1|^2 = c_1\,. \tag{11}$$

From (9), we can then calculate $\hat{u}(s^*)$ which, together with (8), (9) and (11), implies that

$$\hat{\lambda} \int_0^1 \psi_1^2 = c_1 \hat{u}(s^*) = c_1 \hat{\lambda}/\mu^* = \hat{\lambda} \int_0^1 |\psi_1|^2\,.$$

As a result

$$\hat{\lambda}\left(\int_0^1 |\psi_1|^2 - \psi_1^2\right) = 0\,,$$

which implies $\hat{\lambda} = 0$, for otherwise $\operatorname{Im}\psi_1 = \operatorname{Im}\psi_0 = 0$, which is a contradiction. So we conclude that $\hat{\lambda} = 0$. And so we have $\hat{u} = 0$.

We have thus shown (6), and get a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$. □

Now we shall use the Fourier cosine transformation to show the transverality condition and uniqueness of μ^*. If we use $v(x,t) = u(x,t) - c_1 G(x,s)$, the eigenvalue problem is obtained by

$$\lambda v = v_{xx} - (c_1 + b)v - c_1\delta_{s^*} \tag{12}$$

$$\lambda = \mu\big((v^*)'(s^*) + v(s^*)\big) \tag{13}$$

If we take a Fourier cosine transformation in the equation (12), then we have that

$$v(x) = -2c_1 \sum_{k=0}^{\infty} \frac{\cos k\pi s^*}{(k\pi)^2 + c_1 + b + \lambda} \cos k\pi x$$

Furthermore, by using Green's function

$$v(x) = -c_1 G_\lambda(x, s^*). \tag{14}$$

Now, we have the equation (13):

$$\mu\big((v^*)'(s^*) - c_1 G_\lambda(s^*, s^*)\big) = \lambda \tag{15}$$

Here is the main theorem.

THEOREM 3. *For a given pure imaginary eigenvalue $i\beta$, $\beta \neq 0$, there exists a unique μ^* such that $(0, s^*, \mu^*)$ is a Hopf point.*

Proof. We assume that $\beta > 0$ and let $\lambda = i\beta$ in (15), then the real and imaginary parts are obtained by

$$\mu \operatorname{Im}\big((-c_1 G_\beta(s^*, s^*)\big) = \beta \tag{16}$$

$$\mu\big((v^*)'(s^*) + \operatorname{Re}(-c_1 G_\beta(s^*, s^*))\big) = 0 \tag{17}$$

where G_β is Green's function of the operator $A + i\beta$. If we know the existence of β in (17), we may find the value of μ^* corresponding β in (16). Thus, we define

$$T(\beta) = (v^*)'(s^*) + \operatorname{Re}(-c_1 G_\beta(s^*, s^*)).$$

Then

$$\begin{aligned} T(0) &= (v^*)'(s^*) + (-c_1 G(s^*, s^*)) \\ &= \frac{1}{\sqrt{c_1 + b}\sinh\sqrt{c_1 + b}} \Big(1 - \cosh(\sqrt{c_1 + b}(1 - 2s^*))\Big) \\ &< 0 \end{aligned}$$

and $\lim_{\beta\to\infty} T(\beta) = (v^*)'(s^*) > 0$. Furthermore, $T'(\beta) > 0$. Therefore there is a unique β such that $T(\beta) = 0$. From this β, the μ can be uniquely determined from (16).

Now we only need to show the transversality condition. Differentiate with respect to μ in (15) then we have

$$\lambda'(\mu)(1/\mu + c_1 G'_\lambda(s^*, s^*)) = \frac{\lambda}{\mu^2}.$$

Evaluating at $\mu = \mu^*$ (note $\lambda(\mu^*) = i\beta$),

$$\lambda'(\mu^*)\big(\frac{1}{\mu^*} + c_1 G'_\beta(s^*, s^*)\big) = \frac{i\beta}{(\mu^*)^2}.$$

The real part of $\lambda'(\mu^*)$ is

$$\operatorname{Re}(\lambda'(\mu^*)) = \frac{\frac{\beta}{(\mu^*)^2} D}{(C + 1/\mu^*)^2 + D^2},$$

where $C + iD = c_1 G'_\beta(s^*, s^*)$. We only need to examine the sign of D. $D = \mathrm{c}_1 Im G'_\beta(s^*, s^*)$ and

$$D = 4c_1\beta \sum_{k=1}^{\infty} \frac{(\cos k\pi s^*)^2(k^2\pi^2 + c_1 + b)}{((k^2\pi^2 + c_1 + b)^2 + \beta^2)^2}$$

The transversality condition $\mathrm{Re}\lambda'(\mu^*) > 0$ is satisfied. □

Therefore, we have the following theorem for the Hopf bifurcation of (1):

THEOREM 4. *Assume that*

$$\frac{(c_1 - 2a)(c_1 + b) - 2c_1(c_1 + c_2)}{2(c_2 - b)} < \kappa < \frac{(c_1 - 2a)(c_1 + b)}{2(c_2 - b)}$$

so that (1), respectively (2), has a unique stationary solution $(0, s^*)$ *, respectively* (v^*, s^*)*, for all* $\mu > 0$*. Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\widetilde{A} + \mu^* \, Df(0, s^*)$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, \mu^*)$ *is then a Hopf point for (1) and there exists a* C^1*-curve of nontrivial periodic orbits for (1), (2), respectively, bifurcating from* $(0, s^*, \mu^*)$, (v^*, s^*, μ^*)*, respectively.*

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