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THE DIFFERENTIAL PROPERTY OF ODD AND EVEN HYPERPOWER FUNCTIONS

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Abstract. Let $h_e(y)$, $h_o(y)$ denote the limits of the sequences $\{^{2n}x\}$, $\{^{2n+1}x\}$, respectively. From these two functions, we obtain a function $y = p(x)$ as an inverse function of them. Several differential properties of $y = p(x)$ are induced.

1. Introduction

The related problems for hyperpower function

$$h(y) = y^{y^{y^{\cdot^{\cdot^{\cdot}}}}}$$

have been studied by many mathematicians: Euler, Eisenstein, Siedel, etc, because its shape invokes curiosity. Reader can find many information and references (more than one hundred) from the paper of Knoebel [5].

The function $h(y)$ converges for each real y in $[\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\frac{1}{\mathrm{e}}}]$ (see [5]), and diverges elsewhere. Also we can naturally extend the convergence problem from real to complex (see [1]). For each real y in $(0, \mathrm{e}^{-\mathrm{e}})$, the hyperpower function $h(y)$ diverges but we can introduce limit type two functions $h_o(y), h_e(y)$. In order to understand the functions $h_o(y)$ and $h_e(y)$, let's introduce usual notation:

$$^1y = y, \quad ^2y = y^y, \quad ^3y = y^{y^y}, \quad \cdots .$$

Then we can easily imply that for $0 < y < 1$

$$^1y < \ ^3y < \ ^5y < \ ^7y < \cdots \text{ and } \cdots < \ ^8y < \ ^6y < \ ^4y < \ ^2y. \tag{1}$$

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Fig. 1

On the other hand, it is not hard to see that if $x \in (0, e^{-e}]$, then ${}^{2n+1}y < e^{-1} < {}^{2n}y$ (see [5]). Thus we can define two functions $h_o(y) = \lim_{\substack{n\to\infty \\ n \text{ odd}}} {}^{n}y$, $h_e(y) = \lim_{\substack{n\to\infty \\ n \text{ even}}} {}^{n}y$ for y in $(0, \mathrm{e}^{-\mathrm{e}}]$, in fact two functions $h_o(y), h_e(y)$ converge and coincide with the hyperpower function $h(y)$ for each real y in $[\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\frac{1}{\mathrm{e}}}]$. Let's, however, restrict the domain of definition of $h_o(y), h_e(y)$ to $(0, \mathrm{e}^{-\mathrm{e}}]$ for convenience' sake.

The hyperpower function $x = y^{y^{y^{\cdot^{\cdot^{\cdot}}}}}$ induces $x = y^x$, hence the function $y = x^{\frac{1}{x}}$ has the hyperpower function as an inverse function in the interval $[\mathrm{e}^{-1}, \mathrm{e}]$ (see Fig. 1). Also we can find other explicit representation for the inverse functions of the remaining part of $y = x^{\frac{1}{x}}$ in [3]. The functions $h_o(y), h_e(y)$, and $y = x^{\frac{1}{x}}$ satisfy a formula $y^{y^x} = x$ trivially. And the implicit function $y^{y^x} = x$ is composed of above three functions by easy calculation (see [4], [5]). Now we will define a function $y = p(x)$ which is defined on $0 < x < 1$ and exactly composed of two disjoint (but with one common point $(x, y) = (\mathrm{e}^{-1}, \mathrm{e}^{-\mathrm{e}})$) functions

$x = h_o(y)$ and $x = h_e(y)$ (see Fig. 1). The function $y = p(x)$ is our main consideration.

Then the following are established in [4].

i) $h_o(y)$ and $h_e(y)$ are real analytic for $y \in (0, \mathrm{e}^{-\mathrm{e}})$, and $h_o' > 0$ and $h_e' < 0$ for $y \in (0, \mathrm{e}^{-\mathrm{e}})$.

ii) $h_o(y)$ and $h_e(y)$ are continuous for $y \in [0, \mathrm{e}^{-\mathrm{e}}]$, with additional conditions $h_o(0) = 0$, $h_e(0) = 1$, and $h_o(\mathrm{e}^{-\mathrm{e}}) = h_e(\mathrm{e}^{-\mathrm{e}}) = \mathrm{e}^{-1}$.

Hence we know that the function $p(x)$ is continuous for $x \in [0,1]$ and is analytic for $x \in (0, \frac{1}{\mathrm{e}}) \cup (\frac{1}{\mathrm{e}}, 1)$. This paper shows the analyticity at $\frac{1}{\mathrm{e}}$ of $p(x)$. The function $p(x)$ has three singular points $0, 1, \frac{1}{\mathrm{e}}$ and has good differential formula (3) at non-singular points. So our problem is to find the values of $p'(0)$, $p'(1)$, $p'(\frac{1}{\mathrm{e}})$, $p''(0)$, $p''(1)$, $p''(\frac{1}{\mathrm{e}})$, and $p'''(\frac{1}{\mathrm{e}})$. The differential coefficients of $p(x)$ at singular points is not obtained directly, so we have to use indirect methods. In fact, there is a more deep and theoretical approach to the subject for the function $y = p(x)$ (see [2]). However in this paper, we will use methods as elementary as possible.

Now let's see an explicit representation of $p(x)$ from [3], that is

$$p(x) = \begin{cases} x^{(x^{-x})^{(x^{-x})^{\cdot^{\cdot^{\cdot}}}}}, & x \in (0, \dfrac{1}{\mathrm{e}}] \\ (\cdots \log_{(x^{-x})} \log_{(x^{-x})} \mathrm{e})^{-\frac{1}{x}}, & x \in [\dfrac{1}{\mathrm{e}}, 1). \end{cases}$$

2. Analyticity of $p(x)$ at $\frac{1}{\mathrm{e}}$

The function $y = p(x)$ is equal to the $y^{y^x} - x = 0$ with $y \neq x^{\frac{1}{x}}$ from Introduction. By substitution $w = y^x$, the implicit function $y^{y^x} = x$ is changed to a symmetric implicit function $w^w = x^x$ (see Fig. 2) which is divided into two functions $w = x$ and $w = k(x)$. Let's define $q(x, w) = x\log x - w\log w$. We know that $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, so we have

Fig. 2

$$\begin{aligned}\log(a+x) &= \log a + \log(1+\frac{x}{a}) \\ &= \log a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \cdots \quad \text{and} \\ (a+x)\log(a+x) &= a\log a + (1+\log a)x + \frac{x^2}{2a} - \frac{x^3}{6a^2} + \frac{x^4}{12a^3} - \cdots .\end{aligned}$$

Hence we have

$$\begin{aligned}&q(a+x, a+v) \\ &= (x-v)[1+\log a + \frac{x+v}{2a} - \frac{x^2+xv+v^2}{6a^2} + \cdots] \\ &=: (x-v)g_a(x,v).\end{aligned} \tag{2}$$

From the equality between $w^w = x^x$ and $q(x,w)=0$, we know that the functions $w=k(x)$ and $v=k(x+\frac{1}{e})-\frac{1}{e}$ are the same to the implicit functions $g_0(x,w)=0$ and $g_{\frac{1}{e}}(x,v)=0$, respectively. It is trivial that $v^{(n)}(0)=k^{(n)}(\frac{1}{e})$.

Analytic version of implicit function theorem induces the analyticity of two functions $w=k(x)$ and $y=p(x)=(k(x))^{\frac{1}{x}}$ at $x=\frac{1}{e}$. The analytic version of implicit function theorem needs only analytic condition instead of differential condition without change of first derivative condition. In this case, the conditions are satisfied by the following facts that $g_{\frac{1}{e}}(x,v)$ is analytic for two variables x,v, and $\frac{\partial}{\partial v}g_{\frac{1}{e}}(0,0)=\frac{e}{2}\neq 0$. Also we can find another proof of the analyticity of $k(x)$ and $p(x)$ at $\frac{1}{\mathrm{e}}$. In the paper [2], it was proved that two functions $G(x)$ and $H(x)$ are analytic in $(0,1)$. Those functions $G(x)$ and $H(x)$ are, in fact, different representations of $k(x)$ and $p(x)$, respectively.

Anyway we conclude that the function $y=p(x)$ is continuous on $[0,1]$ and is analytic on $(0,1)$.

3. The values of $p'(0)$, $p'(1)$, and $p'(\frac{1}{\mathrm{e}})$

The function $y=p(x)$ satisfies $y^{y^x}=x$ and $y\neq x^{\frac{1}{x}}$, then we can induce $p'(x)$ by using $z=y^{y^x}-x$ and $\frac{\partial z}{\partial x}=y^{y^x}\cdot y^x(\log y)^2-1$, $\frac{\partial z}{\partial y}=y^{y^x}\cdot y^{x-1}(1+x\log y)$, so

$$p'(x)=-\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y}=\frac{y-x\cdot y^{x+1}(\log y)^2}{x\cdot y^x(1+\log y^x)}. \tag{3}$$

Hence $p'(x)$ at a given point on the graph $y=p(x)$ is obtained by above formula, but $p'(0),p'(1)$, and $p'(\frac{1}{\mathrm{e}})$ are not obtained from the formula. In order to get the differential coefficients at $0,1,\frac{1}{\mathrm{e}}$, we will mainly use the following squeezing lemma. The proof is trivial.

Lemma 1. *Let f,f_1,f_2 be continuous and $f_1\leq f\leq f_2$ on (a,b) and f_1,f_2 be differentiable at c, and suppose that $f_1(c)=f_2(c)$ and $f_1'(c)=f_2'(c)$ for some $c\in(a,b)$. Then f is differentiable at c and $f'(c)=f_1'(c)=f_2'(c)$.*

We already know that the function $y^{y^x}=x$ can be transformed into $w^w=x^x$ by letting $w=y^x$. The $\pm$ signs in Fig. 2 means positiveness or negativeness of w^w-x^x at that place, and will be used at the proof of Proposition 7.

Proposition 2. *$h'_e(0) = -\infty$, so $p'(1) = 0$.*

Proof. From the relation (1), we know ${}^{\text{even}}y < {}^{2}y$ for $y \in (0,1)$. Since $\lim_{y\to 0} y^y = 1$, we set $y^y(0) := 1$, and by the Mean Value Theorem, $\lim_{y\to 0}(y^y)' = -\infty$ induces $(y^y)'(0) = -\infty$. Hence the function $x = h_e(y)$ is squeezed by $y = 0$ and $x = y^y$ at $(x, y) = (1, 0)$, and consequently $h'_e(0) = -\infty$ by the squeezing lemma. Then $p'(1) = 0$ is trivial. □

Proposition 3. *$p'(\frac{1}{\mathrm{e}}) = 0$.*

Proof. In section 2, we know that $k'(\frac{1}{e})$ is the same to the differential coefficient $v'(0)$ induced by the implicit function $g_{\frac{1}{e}}(x, v) = 0$. So we have $\frac{\partial}{\partial x} g + \frac{\partial}{\partial v} g \cdot \frac{dv}{dx} = 0$ and $\frac{dv}{dx}|_{x=0} = -g_x(0,0)/g_v(0,0) = -1$. Hence $k'(\frac{1}{e}) = -1$. Finally $p'(\frac{1}{e}) = 0$ is deduced from $p(x) = k(x)^{\frac{1}{x}}$. □

The remaining thing is to get $p'(0)$, this calculation is more difficult than $p'(1)$, and $p'(\frac{1}{\mathrm{e}})$. We need the following lemma.

Lemma 4. *Let f be continuous and differentiable on $(0, \infty)$. If $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} x^{f(x)} = 1$, and $\lim_{x\to 0} \sqrt{x} f'(x)$ exists with finite value. Then $\lim_{x\to 0} \frac{x^{x^{f(x)}}}{x} = 1$.*

Proof. Since $\frac{x^{x^f}}{x} = \mathrm{e}^{(x^f - 1)\log x}$, we need $\lim_{x\to 0} \frac{x^f - 1}{(\log x)^{-1}} = 0$. By L'Hôpital's law, we have

$$\begin{aligned}
\lim_{x\to 0} \frac{x^f - 1}{(\log x)^{-1}} &= \lim_{x\to 0} \frac{x^f(f'\log x + f \cdot \frac{1}{x})}{-(\log x)^{-2} \cdot \frac{1}{x}} \\
&= \lim_{x\to 0} -x^f(f' \cdot x(\log x)^3 + f \cdot (\log x)^2) \\
&= \lim_{x\to 0} -x^f(\sqrt{x} f' \cdot \sqrt{x}(\log x)^3 + \frac{f}{\sqrt{x}} \cdot \sqrt{x}(\log x)^2).
\end{aligned}$$

It is easy that $\lim_{x\to 0} \sqrt{x}(\log x)^3 = \lim_{x\to 0} \sqrt{x}(\log x)^2 = 0$. Also L'Hôpital's law deduces $\lim_{x\to 0} \frac{f}{\sqrt{x}} = \lim_{x\to 0} 2\sqrt{x} f'$, so the two limits take the same finite value. Hence we can conclude
$\lim_{x\to 0}(x^f - 1)\log x = 0$. □

Since The function $f(x) = x$ satisfies the three conditions of Lemma 4, we have the following property;

Corollary 5. $\lim_{x\to 0} \frac{x^{x^x}}{x} = 1.$

Corollary 6. $\lim_{x\to 0} \frac{x^{x(x^x-1)}}{x} = 1.$

Proof. Let $f(x) = x^x - 1$, then $\lim_{x\to 0} f(x) = 0$ is trivially checked, and Corollary 5 induces $\lim_{x\to 0} x^f = 1$, and easily $\lim_{x\to 0} \sqrt{x} f' = \lim_{x\to 0} x^x \sqrt{x}(1 + \log x) = 0$. Therefore all conditions of the lemma are satisfied. $\square$

Let's prove the $p'(0) = 1$.

Proposition 7. $p'(0) = 1.$

Proof. The positiveness of $x^{x^x} - x$ is obtained by the formula (1) for $x \in (0, 1)$ and trivially for $x \in (1, \infty)$. So we have $x^{x^x} - x \geq 0$ (equals when $x = 1$) for all positive x. Therefore so $y = x$ is an upper squeezing function of $y = p(x)$ at $x = 0$.

A function $w = x^{x^{x^x}} = {}^4x$ satisfy $w^w - x^x < 0$ for $x \in (0, \delta)$ (sufficiently small δ), because we get $w^w - x^x = x^{x({}^3x + {}^2x)} - x^x$ and ${}^3x + {}^2x < 1$ for $x \in (0, \delta)$, which is induced from $\lim_{x\to 0}({}^3x + {}^2x) = 1$ and $\lim_{x\to 0}({}^3x + {}^2x)' = -\infty$. So $w = {}^4x$ lies on the lower of the decreasing part of $w^w = x^x$ around the zero. The fact that $0 < f_1 < f_2$ implies $f_1^{\frac{1}{x}} < f_2^{\frac{1}{x}}$ for positive x induces that $y = w^{\frac{1}{x}} = ({}^4x)^{\frac{1}{x}}$ is located in the lower part of $y = p(x)$ when x is near 0. We have to show that if $y = x^{\frac{x^{x^x}}{x}}$, then $y'(0) = 1$. By supposing $y(0) = \lim_{x\to 0} y(x)$ we have $y(0) = 0$ by corollary 6 or $0 \leq y(x) \leq p(x)$ with $p(0) = 0$. Then it can be determined that $y'(0) = \lim_{\epsilon\to 0} \frac{x^{x(x^x-1)} - 0}{x} = 1$ by corollary 6. Therefore we can conclude $p'(0) = 1$ by the two squeezing functions $y = x$ and $y = x^{\frac{x^{x^x}}{x}}$ and the squeezing lemma. $\square$

Also we can find other methods from [2] for finding of values $p'(0)$, $p'(1)$, and $p'(\frac{1}{\mathrm{e}})$. The formula (2.5) and Proposition 4.4 in [2] show the methods.

We know that $\lim_{y\to 0+} h_o'(y) = 1)$ from Proposition 2.5 in [2]. This implies $\lim_{x\to 0+} p'(x) = 1$. And the formulas $h_e(y) = y^{h_o(y)}$ and $\lim_{y\to 0+} h_o'(y) = 1$ induces $\lim_{y\to 0+} h_e'(y) = -\infty$ and $\lim_{x\to 1-} p'(x) = 0$.

Hence the function $p(x)$ is C^1-function on $[0, 1]$.

4. The values of $p''(0)$, $p''(\frac{1}{\mathrm{e}})$, $p''(1)$, and $p'''(\frac{1}{\mathrm{e}})$

Now we can induce the second or third derivative of $p(x)$ at singular points. In particular, the evaluation of $p''(0)$ is more difficult than others, so we need a result of paper [2]. In this section, the variable w denotes $k(x)$ and we simplify $g_{\frac{1}{e}}(x,v)$ as $g(x,v)$ for convenience' sake.

PROPOSITION 8. *$k''(\frac{1}{e}) = \frac{2}{3}e$.*

Proof. The value $k''(\frac{1}{e})$ is obtained from $g(x,v(x)) = 0$ and we can extract the value of $v''(0)$ from it. We induce that $g_x + g_v v' = 0$ and $g_{xx} + (2g_{xv} + g_{vv}v')v' + g_v v'' = 0$ from $\frac{d}{dx}g(x,v(x)) = 0$ and $\frac{d^2}{dx^2}g(x,v(x)) = 0$, so

$$v'' = -\frac{g_{xx} + (2g_{xv} + g_{vv}v')v'}{g_v} = \frac{2g_{xv} - g_{xx} - g_{vv}}{g_v}.$$

The values $g_v(0,0) = \frac{1}{2a}$, $g_{xv}(0,0) = -\frac{1}{6a^2}$, and $g_{xx}(0,0) = g_{vv}(0,0) = -\frac{1}{3a^2}$ are evaluated by the formula (2). Hence $v''(0) = \frac{2}{3a} = \frac{2e}{3}$. $\square$

PROPOSITION 9. *$p''(\frac{1}{e}) = -\frac{1}{3}e^{3-e}$.*

Proof. We know $y = p(x) = k(x)^{\frac{1}{x}} = w^{\frac{1}{x}}$.
So

$$y' = w^{\frac{1}{x}}\frac{w' \cdot x - w\log w}{wx^2} = w^{\frac{1}{x}}\frac{w' \cdot x - x\log x}{wx^2} = w^{\frac{1}{x}}\frac{w' - \log x}{wx},$$

and more

$$\begin{aligned} y'' &= (y\frac{w' - \log x}{wx})' \\ &= y'\frac{w' - \log x}{wx} + y\frac{(w'' - \frac{1}{x})wx - (w' - \log x)(w + w'x)}{w^2x^2}. \end{aligned}$$

Already we get $y'(\frac{1}{e}) = 0$ by Proposition 3, $w''(\frac{1}{e}) = \frac{2e}{3}$ by Proposition 8, and $w(\frac{1}{e}) = \frac{1}{e}$, $w'(\frac{1}{e}) = -1$, $y(\frac{1}{e}) = e^{-e}$. Therefore $y''(\frac{1}{e}) = -\frac{1}{3}e^{3-e}$. $\square$

PROPOSITION 10. *$k'''(\frac{1}{e}) = -\frac{2}{3}e^2$.*

Proof. The function $w = k(x)$ is symmetric to the line $w = x$, so its inverse function is $x = k(w)$. Inverse function theorem says $(f^{-1})'(y) = \frac{1}{f'(x)}$, and this formula makes

$$(f^{-1})'''(y) = \frac{3(f''(x))^2 - f'''(x) \cdot f'(x)}{(f'(x))^5}.$$

So we get

$$k'''(\frac{1}{e}) = \frac{3(k''(\frac{1}{e}))^2 - k'''(\frac{1}{e}) \cdot k'(\frac{1}{e})}{(k'(\frac{1}{e}))^5}.$$

Hence $k'''(\frac{1}{e}) = -\frac{2}{3}e^2$. □

PROPOSITION 11. $p'''(\frac{1}{e}) = \frac{1}{3}e^{4-e}$.

Proof. The function $y = w^{\frac{1}{x}}$ induces $y' = y\frac{w' - \log x}{wx} =: y \cdot \alpha(x)$. By $\alpha(\frac{1}{e}) = 0$ and $y'(\frac{1}{e}) = 0$, we get $y'''(\frac{1}{e}) = y(\frac{1}{e})\alpha''(\frac{1}{e})$. Let's define $l(x) := w' - \log x$ and $m(x) := w \cdot x$, then $\alpha(x) = \frac{l(x)}{m(x)}$. By $l(\frac{1}{e}) = 0$ and $m'(\frac{1}{e}) = 0$, we have $\alpha''(\frac{1}{e}) = \frac{l''(\frac{1}{e})}{m(\frac{1}{e})}$. So

$$y'''(\frac{1}{e}) = y(\frac{1}{e})\frac{l''(\frac{1}{e})}{m(\frac{1}{e})} = \frac{y(\frac{1}{e})}{m(\frac{1}{e})}(w'''(\frac{1}{e}) + e^2) = \frac{1}{3}e^{4-e}.$$

□

For $n \geq 4$, the values of $k^{(n)}(\frac{1}{e})$ and $p^{(n)}(\frac{1}{e})$ also can be deduced by essentially the formula (2) too. But the author cannot find the exact or inductive representations of them.

Now we evaluate $p''(0)$ and $p''(1)$.

PROPOSITION 12. $p''(0) = -\infty$.

Proof. Proposition 2.5 in [2] say $h''_o(0) = \infty$. We know $(f^{-1})''(y) = -\frac{f''(x)}{(f'(x))^3}$, so we get $p''(0) = -\infty$. □

LEMMA 13. $\lim_{x \to 1-}(x-1)log w = 0$.

Proof. We use L'Hôpital's law and $w' = \frac{1+\log x}{1+\log w}$. Then we have

$$\lim_{x\to 1-} \frac{x-1}{(\log w)^{-1}} = \lim_{x\to 1-} -\frac{w(\log w)^2}{w'}$$
$$= \lim_{x\to 1-} -\frac{w(\log w)^2 + w(\log w)^3}{1+\log x} = 0.$$

□

Lemma 14. $\lim_{x\to 1-} \frac{w^{\frac{1}{x}}}{w} = 1$.

Proof. Since $\lim_{x\to 1-} w^{x-1} = 1$ (by Lemma 13) and

$$\lim_{x\to 1-} \left(\frac{1}{1-\epsilon}\right)^x = \frac{1}{1-\epsilon} > 1 \text{ for fixed } \epsilon \text{ with } 0 < \epsilon < 1,$$

we get that $w^{x-1} < (\frac{1}{1-\epsilon})^x$ is satisfied on $(\delta, 1)$ for some δ (depending on ϵ) with $0 < \delta < 1$. We know that $(1-\epsilon)w < w^{\frac{1}{x}}$ and $x < 1$ implies $w^{\frac{1}{x}} < w$. Hence we conclude that $1 - \epsilon < \frac{w^{\frac{1}{x}}}{w} < 1$ for $x \in (\delta, 1)$. By successive taking $x \to 1-$ and $\epsilon \to 0$ at three sides of the inequality, we obtain the result. □

Proposition 15. $p''(1) = \infty$.

Proof. From the formula (3), we know $p'(x) = p(x)\frac{1-\log x \log w}{xw(1+\log w)}$. So

$$p''(x) = \lim_{x\to 1-} \frac{p'(x)}{x-1} = \lim_{x\to 1-} \frac{p(x)(1-\log x\log w)}{(x-1)xw(1+\log w)}.$$

We know that $\lim_{x\to 1-} \log x \log w = 0$ and $\lim_{x\to 1-}(x-1)\log w = 0$ and $\lim_{x\to 1-} \frac{p(x)}{w} = 1$ by Lemma 4.2 in [2] and Lemma 13 and Lemma 14, respectively. Therefore we have $p''(1) = \infty$. □

Proposition 2.5 and Lemma 4.2 in [2] are used in the proofs of Proposition 12, 15, respectively. Lemma 4.2 is elementary, so Proposition 15 is eventually elementary, but Proposition 12 is not.

The author left several unsolved questions for readers. Other unsolved questions related the subject can be found in [2].

Q1. Find the exact or inductive representations for the values of $k^{(n)}(\frac{1}{e})$ and $p^{(n)}(\frac{1}{e})$.

Q2. Prove that the function $y = p(x)$ has a unique inflection point.

Q3. Prove that the derivative function $y = p'(x)$ is strictly convex.

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