Kangweon-Kyungki Math. Jour. **12** (2004), No. 1, pp. 91–106

C^∞-REGULARITY OF INTERFACE OF SOME ONE-DIMENSIONAL NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract. We prove the regularity of a moving interface of the solutions of the initial value problem of equation (1.1) is C^∞.

1. Introduction

We consider the Cauchy problem of the form

$$u_t = \frac{\partial}{\partial x}\left(\frac{\partial u^m}{\partial x}\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\right) \quad \text{in} \quad S = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+\} \tag{1.1}$$

where $m > 0$, $p > 1 + \frac{1}{m}$.

Equations like (1.1) were studied many authors and arise in different physical situations, for the detail see [3]. An important quantity of the study of equation (1.1) is the local velocity of propagation $V = -v_x|v_x|^{p-2}$, whose expression in terms of u can be obtained by writing the equation as a conservation law in the form

$$u_t + (uV)_x = 0.$$

In this way we get

$$V = -v_x|v_x|^{p-2},$$

where the nonlinear potential $v(x,t)$ is

$$v = \frac{m(p-1)}{m(p-1)-1} u^{m-\frac{1}{p-1}} \tag{1.2}$$

and by a direct computation v satisfies

$$v_t = (m(p-1)-1)v|v_x|^{p-2}v_{xx} + |v_x|^p. \tag{1.3}$$

Received February 28, 2004.
2000 Mathematics Subject Classification: 35J.
Key words and phrases: free boundary, C^∞ regularity.

In [3], it was shown that V satisfies

$$V_x \le \frac{1}{(p-1)(m+1)t},$$

which can also be written as

$$(v_x|v_x|^{p-2})_x \ge -\frac{1}{(p-1)(m+1)t} \tag{1.4}$$

Without loss of generality, we may consider the case where u_0 vanishes on $\mathbb{R}^-$ and is a continuous positive function, at least, on an interval $(0,a)$ with $a>0$. Let

$$P[u] = \{(x,t) \in S : u(x,t) > 0\}$$

be the positivity set of a solution u. Then $P[u]$ is bounded to the left in (x,t)-plane by the left interface curve $x=\zeta(t)$[3], where

$$\zeta(t) = \inf\{x \in \mathbb{R} : u(x,t) > 0\}.$$

Moreover there is a time $t^* \in [0,\infty)$, called the waiting time, such that $\zeta(t)=0$ for $0 \le t \le t^*$ and $\zeta(t) < 0$ for $t > t^*$. It is shown [3] that t^* is finite(possibly zero) and $\zeta(t)$ is a nonincreasing C^1 function on (t^*,∞). Actually it is shown that $\zeta'(t) < 0$ for every $t > t^*$, i.e., a moving interface never stop.

For the interface of the porous medium equation

$$\begin{cases} u_t = \triangle(u^m) & \text{in} \quad \mathbb{R}^n \times [0,\infty), \\ u(x,0) = u_0 & \text{on} \quad \mathbb{R}^n \end{cases}$$

much more is known. D. G. Aronson and J. L. Vazquez [2] showed the interfaces are smooth after the waiting time. S. Angenent [1] showed that the interfaces are real analytic after the waiting time.

On the other hand much less is known for the equation (1.1). For dimensions $n \ge 2$, Zhao Junning [6] showed, under some nondegeneracy conditions on the initial data, the interface is Lipschitz continuous and we [4] improved this result, showing that, under the same hypotheses, the interface is a $C^{1,\alpha}$ surface after some time.

In this paper we show the interfaces of the solutions of (1.1) are smooth after the waiting time. In establishing C^∞ regularity of the interfaces, we follow the ideas of Aronson and Vazquez. They showed the C^∞ regularity by establishing the bounds for $v^{(k)}$ for $k \ge 2$, where $v = \frac{m}{m-1}u^{m-1}$ represents the pressure of the gas flow through a porous medium, while u represents the density.

2. The Upper and Lower Bound for v_{xx}

Let $q = (x_0, t_0)$ be a point on the left interface, so that $x_0 = \zeta(t_0)$, $v(x, t_0) = 0$ for all $x \leq \zeta(t_0)$, and $v(x, t_0) > 0$ for all sufficiently small $x > \zeta(t_0)$. We assume the left interface is moving at q. Thus $t_0 > t^*$. We shall use the notation

$$R_{\delta,\eta} = R_{\delta,\eta}(t_0) = \{(x, t) \in \mathbb{R}^2 : \zeta(t) < x \leq \zeta(t) + \delta, t_0 - \eta \leq t \leq t_0 + \eta\}.$$

Proposition 2.1. *Let q be the point as above. Then there exist positive constants C, δ and η depending only on p, q, m and u such that*

$$v_{xx} \geq C \quad \text{in} \quad R_{\delta,\eta/2}.$$

Proof. From (1.4) we have, $v_{xx} \geq -\dfrac{1}{(m+1)(p-1)^2|v_x|^{p-2}t}$. But from Lemma 4.4 in [3] v_x is bounded away and above from zero near the interface where $u(x, t) > 0$. □

Proposition 2.2. *Let $q = (x_0, t_0)$ be as before. Then there exist positive constants C_2,δ and η depending only on p, q and u such that*

$$v_{xx} \leq C_2 \quad \text{in} \quad R_{\delta,\eta/2}.$$

Proof. From Theorem 2 and Lemma 4.4 in [3] we have

$$\zeta'(t_0) = -v_x|v_x|^{p-2} = -v_x^{p-1} = -a \tag{2.1}$$

and

$$v_t = |v_x|^p \tag{2.2}$$

on the moving part of the interface $\{x = \zeta(t), t > t^*\}$. Choose $\epsilon > 0$ satisfying

$$(p-1)a - [4m(p-1) + p - 2]\epsilon \geq 2\mu(a + \epsilon)\epsilon \tag{2.3}$$

and

$$(a - \epsilon)^{\frac{1}{p-1}} \geq 2|p - 3|(a + \epsilon)^{\frac{1}{p-1}}\epsilon \tag{2.4}$$

where $\mu = 2\{M(2p-3) + p(p-1)\}$. Then by Theorem 2 in [3], there exists a $\delta = \delta(\epsilon) > 0$ and $\eta = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_{\delta,\eta} \subset P[u]$,

$$(a - \epsilon)^{\frac{1}{p-1}} < v_x < (a + \epsilon)^{\frac{1}{p-1}} \tag{2.5}$$

and

$$vv_{xx} \leq (a - \epsilon)^{\frac{2}{p-1}}\epsilon \tag{2.6}$$

in $R_{\delta,\eta}$. Then we have

$$(a-\epsilon)^{\frac{1}{p-1}}(x-\zeta) < v(x,t) < (a+\epsilon)^{\frac{1}{p-1}}(x-\zeta) \tag{2.7}$$

in $R_{\delta,\eta}$ and

$$-(a+\epsilon) < \zeta'(t) < -(a-\epsilon) \quad \text{in} \quad [t_1, t_2] \tag{2.8}$$

where $t_1 = t_0 - \eta$ and $t_2 = t_0 + \eta$. We set

$$\zeta^*(t) = \zeta(t_1) - b(t - t_1) \tag{2.9}$$

where $b = a + 2\epsilon$. Then clearly $\zeta(t) > \zeta^*(t)$ in $(t_1, t_2]$.

Next, set $M = m(p-1) - 1$. Then on $P[u]$, $w \equiv v_{xx}$ satisfies

$$\begin{aligned} L(w) &= w_t - Mv|v_x|^{p-2}w_{xx} - 3(p-2)Mv|v_x|^{p-4}v_x w w_x \\ &\quad -\{2M+p\}|v_x|^{p-2}v_x w_x - \{M(2p-3)+p(p-1)\}|v_x|^{p-2}w^2 \\ &\quad -(p-2)M(p-3)v|v_x|^{p-4}w^3. \end{aligned}$$

We shall construct a barrier for w in $R_{\delta,\eta}$ of the form

$$\phi(x,t) \equiv \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)},$$

where α and β will be decided later.

By a direct computation, we have

$$\begin{aligned} L(\phi) &= \frac{\alpha}{(x-\zeta)^2}\Big\{\zeta' - Mv|v_x|^{p-2}\frac{2}{x-\zeta} + [2M+p]|v_x|^{p-2}v_x\Big\} \\ &\quad + \frac{\beta}{(x-\zeta^*)^2}\Big\{\zeta^{*\prime} - Mv|v_x|^{p-2}\frac{2}{x-\zeta^*} + [2M+p]|v_x|^{p-2}v_x\Big\} \\ &\quad -[M(2p-3)+p(p-1)]|v_x|^{p-2}\phi^2 + G \end{aligned}$$

where

$$\begin{aligned} G &= -3(p-2)Mv|v_x|^{p-4}v_x\phi\phi_x - (p-2)M(p-3)v|v_x|^{p-4}\phi^3 \\ &= (p-2)Mv|v_x|^{p-4} \times \\ &\quad \phi\left(3v_x[\frac{\alpha}{(x-\zeta)^2} + \frac{\beta}{(x-\zeta^*)^2}] - (p-3)[\frac{\alpha}{x-\zeta} + \frac{\beta}{x-\zeta*}]^2\right). \end{aligned}$$

If we choose α and β satisfying

$$v_x \geq |p-3| \max(\alpha, \beta)$$

then $G \geq 0$ in $R_{\delta,\eta}$. Now set $\bar{A} = \dfrac{\alpha}{(x-\zeta)^2}$ and $\bar{B} = \dfrac{\beta}{(x-\zeta^*)^2}$. Then we have

$$\begin{aligned} L(\phi) \;&\geq\; \bar{A}\left\{(p-1)a - [4m(p-1)+p-3]\epsilon - \mu(a+\epsilon)^{\frac{p-2}{p-1}}\alpha\right\} \\ &\quad + \bar{B}\left\{(p-1)a - [4m(p-1)+p-2]\epsilon - \mu(a+\epsilon)^{\frac{p-2}{p-1}}\beta\right\} \end{aligned}$$

where μ is as before. Set

$$0 < \alpha \leq \min\{\frac{(a-\epsilon)^{\frac{1}{p-1}}}{|p-3|}, \frac{(p-1)a - [4m(p-1)+p-3]\epsilon}{\mu(a+\epsilon)^{\frac{p-2}{p-1}}}\} = \alpha_0$$

and

$$\beta = \min\{\frac{(a-\epsilon)^{\frac{1}{p-1}}}{|p-3|}, \frac{(p-1)a - [4m(p-1)+p-2]\epsilon}{\mu(a+\epsilon)^{\frac{p-2}{p-1}}}\}. \tag{2.10}$$

Then $L(\phi) \geq 0$ in $R_{\delta,\eta}$ for all $\alpha \in (0, \alpha_0]$ and β.

Let us now compare w and ϕ on the parabolic boundary of $R_{\delta,\eta}$. In view of (2.6) and (2.7) we have

$$v_{xx} < \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{x-\zeta} \quad \text{in} \quad R_{\delta,\eta}$$

and in particular

$$v_{xx}(\zeta(t)+\delta, t) \leq \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{\delta} \quad \text{in} \quad [t_1, t_2].$$

By the mean value theorem and (2.8) we have for some $\tau \in (t_1, t_2)$

$$\begin{aligned} \zeta(t) + \delta - \zeta^*(t) \;&=\; \delta + (a+2\epsilon)(t-t_1) + \zeta'(\tau)(t-t_1) \\ &\leq\; \delta + 3\epsilon(t-t_1) \leq \delta + 6\epsilon\eta. \end{aligned}$$

Now set

$$\eta \equiv \min\{\eta(\epsilon), \delta(\epsilon)/6\epsilon\}.$$

Since ϵ satisfies (2.3), (2.4) and $\beta \leq \alpha_0$ it follows that

$$\phi(\zeta+\delta, t) \geq \frac{\beta}{2\delta} \geq \frac{(a+\epsilon)^{\frac{1}{p-1}}}{\delta}\epsilon \geq v_{xx} \quad \text{on} \quad [t_1, t_2].$$

Moreover

$$\phi(x, t_1) \geq \frac{\beta}{x-\zeta(t_1)} > \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{x-\zeta(t_1)} > v_{xx}(x, t_1) \quad \text{on} \quad (\zeta(t_1), \zeta(t_1)+\delta].$$

Let $\Gamma = \{(x,t) \in \mathbb{R}^2 : x = \zeta(t), t_1 \le t \le t_2\}$. Clearly Γ is a compact subset of $\mathbb{R}^2$. Fix $\alpha \in (0, \alpha_0)$. For each point $s \in \Gamma$ there is an open ball B_s centered at s such that

$$(vv_{xx})(x,t) \le \alpha(a-\epsilon)^{\frac{1}{p-1}} \quad \text{in} \quad B_s \cap P[u].$$

In view of (2.7) we have

$$\phi(x,t) \ge \frac{\alpha}{x-\zeta} \ge v_{xx}(x,t) \quad \text{in} \quad B_s \cap P[u].$$

Since Γ can be covered by a finite number of these balls it follows that there is a $\gamma = \gamma(\alpha) \in (0, \delta)$ such that

$$\phi(x,t) \ge w(x,t) \quad \text{in} \quad R_{\gamma,\eta}.$$

Thus for every $\alpha \in (0, \alpha_0)$, ϕ is a barrier for w in $R_{\delta,\eta}$. By the comparison principle for parabolic equations [5] we conclude that

$$v_{xx}(x,t) \le \frac{\alpha}{x-\zeta} + \frac{\beta}{x-\zeta^*} \quad \text{in} \quad R_{\delta,\eta},$$

where β is given by (2.10) and $\alpha \in (0, \alpha_0)$ is arbitrary. Now let $\alpha \downarrow 0$ to obtain

$$v_{xx}(x,t) \le \frac{\beta}{x-\zeta^*} \le \frac{2\beta}{\epsilon\eta} \quad \text{in} \quad R.$$

□

3. Bounds for $\left(\frac{\partial}{\partial x}\right)^3 v$

In this section we find the estimates of $v^{(3)} \equiv \left(\frac{\partial}{\partial x}\right)^3 v$. By a direct computation we have,

$$\begin{aligned} L_3(v^{(3)}) &= v_t^{(3)} - Mvv_x^{p-2}v_{xx}^{(3)} - (A+B)v_x^{(3)} - Cv^{(3)} - D(v^{(3)})^2 \\ &\quad -Ev_x^{p-3}v_{xx}^3 - M(p-2)(p-3)(p-4)vv_x^{p-5}v_{xx}^4 = 0 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
A &= Mv_x^{p-1} + M(p-2)vv_x^{p-3}v_{xx}, \\
B &= (2M+p)v_x^{p-1} + 3M(p-2)vv_x^{p-3}v_{xx}, \\
C &= v_{xx}v_x^{p-2}\{(2M+p)(p-1) + 2[M(2p-3)+p(p-1)] \\
&\quad +6M(p-2)(p-3)vv_x^{-2}v_{xx} + 3M(p-2)\}, \\
D &= 3M(p-2)vv_x^{p-3}
\end{aligned}$$

and

$$E = [M(2p-3)+p(p-1)](p-2) + M(p-2)(p-3).$$

Suppose that $q = (x_0, t_0)$ is a point on the left interface for which (2.1) holds. Fix $\epsilon \in (0, a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_0 \equiv R_{\delta_0,\eta_0}(t_0) \subset P[u]$ and (2.6) holds. Thus we also have (2.7) and (2.8) in R_0. Then by rescaling and interior estimate we have

Proposition 3.1. *There are constants $K \in \mathbb{R}^+$, $\delta \in (0, \delta_0)$, and $\eta \in (0, \eta_0)$ depending only on m, p, q and C_2 such that*

$$|v^{(3)}(x,t)| \le \frac{K}{x - \zeta(t)} \quad \text{in} \quad R_{\delta,\eta}.$$

Proof. Set

$$\delta = \min\{\frac{2\delta_0}{3}, 2s\eta_0\}, \qquad \eta = \eta_0 - \frac{\delta}{4s},$$

and define

$$R(\overline{x}, \overline{t}) \equiv \left\{(x,t) \in \mathbb{R}^2 : |x - \overline{x}| < \frac{\lambda}{2}, \overline{t} - \frac{\lambda}{4s} < t \le \overline{t}\right\}$$

for $(\overline{x}, \overline{t}) \in R_{\delta,\eta}$, where $s = a + \epsilon$ and $\lambda = \overline{x} - \zeta(\overline{t})$. Then $(\overline{x}, \overline{t}) \in R_{\delta,\eta}$ implies that $R(\overline{x}, \overline{t}) \subset R_0$. Since $\delta_0 \ge \frac{3\delta}{2}$, $\lambda < \delta$ and ζ is nonincreasing, we have

$$t_0 - \eta_0 = t_0 - \eta - \frac{\lambda}{4s} < t < t_0 + \eta < t_0 + \eta_0$$

and

$$\overline{x} - \frac{\lambda}{2} = \overline{x} - \frac{\overline{x} + \zeta(\overline{t})}{2} = \frac{\overline{x} + \zeta(\overline{t})}{2} > \zeta(t_0 + \eta_0)$$

$$\zeta(t_0 - \eta) + \delta + \frac{\lambda}{2} < \zeta(t_0 - \eta_0).$$

Also observe that for each $(\overline{x},\overline{t}) \in R_{\delta,\eta}$, $R(\overline{x},\overline{t})$ lies to the right of the line $x = \zeta(\overline{t})+s(\overline{t}-t)$. Next set $x = \lambda\xi+\overline{x}$ and $t = \lambda\tau+\overline{t}$. The function

$$W(\xi,\tau) \equiv v_{xx}(\lambda\xi+\overline{x},\lambda\tau+\overline{t}) = v_{xx}(x,t)$$

satisfies the equation

$$\begin{aligned} W_\tau &= \left\{M\frac{v}{\lambda}v_x^{p-2}W_\xi+(2M+p)v_x^{p-1}W\right\}_\xi \\ &\quad +[2M(p-2)vv_x^{p-3}v_{xx}-Mv_x^{p-1}]W_\xi \\ &\quad +\lambda[M(p-2)(p-3)vv_x^{p-4}(v_{xx})^3-Mv_x^{p-2}(v_{xx})^2] \end{aligned} \tag{3.2}$$

in the region

$$B \equiv \left\{(\xi,\tau)\in\mathbb{R}^2 : |\xi| \le \frac{1}{2}, -\frac{1}{4s} < \tau \le 0\right\},$$

and $|W| \le C_2$ in B. In view of (2.7) and (2.8)

$$(a-\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda} \le \frac{v(x,t)}{\lambda} \le (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda}$$

and

$$\zeta(\overline{t}) \le \zeta(t) \le \zeta(\overline{t})+s(\overline{t}-t) \le \zeta(\overline{t})+\frac{\lambda}{4}.$$

Therefore

$$\frac{\lambda}{4} = \overline{x}-\frac{\lambda}{2}-\zeta(\overline{t})-\frac{\lambda}{4} \le x-\zeta(t) \le \overline{x}+\frac{\lambda}{2}-\zeta(\overline{t}) = \frac{3\lambda}{2}$$

which implies

$$\frac{(a-\epsilon)^{\frac{1}{p-1}}}{4} \le \frac{v}{\lambda} \le \frac{3(a+\epsilon)^{\frac{1}{p-1}}}{2}.$$

Hence by (2.5) equation (3.2) is uniformly parabolic in B. Moreover, it follows from Proposition 2.2 that W satisfies all of the hypotheses of Theorem 5.3.1 of [5]. Thus we conclude that there exists a constant $K = K(a,m,p,C_2) > 0$ such that

$$\left|\frac{\partial}{\partial\xi}W(0,0)\right| \le K;$$

that is,

$$|v^{(3)}(\overline{x},\overline{t})| \le \frac{K}{\lambda}.$$

Since $(\overline{x},\overline{t}) \in R_{\delta,\eta}$ is arbitrary, this proves the proposition. □

We now turn to the barrier construction. If $\gamma \in (0,\delta)$ we will use the notation

$R^\gamma_{\delta,\eta} = R^\gamma_{\delta,\eta}(t_0) \equiv \{(x,t) \in \mathbb{R}^2 : \zeta(t)+\gamma \le x \le \zeta(t)+\delta, t_0-\eta \le t \le t_0+\eta\}.$

PROPOSITION 3.2. *Let R_{δ_1,η_1} be the region constructed in the proof of Proposition 2.2 with*

$$0 < \delta_1 < \frac{(p-1)a^{\frac{1}{p-1}}}{12M(p-2)K}. \tag{3.3}$$

For $(x,t) \in R^\gamma_{\delta_1,\eta_1}$, let

$$\phi_\gamma(x,t) \equiv \frac{\alpha}{x-\zeta(t)-\gamma/3} + \frac{\beta}{x-\zeta^*(t)} \tag{3.4}$$

where ζ^* is given by (2.9), and α and β are positive constant less than $K/2$. Then there exist $\delta \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ depending only on a, m, p and C_2 such that

$$L_3(\phi_\gamma) \ge 0 \quad \textit{in} \quad R^\gamma_{\delta,\eta}$$

for all $\gamma \in (0,\delta)$.

Proof. Choose ϵ such that

$$0 < \epsilon < \frac{(p-1)a}{13p-23}. \tag{3.5}$$

There exist $\delta_2 \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ such that (2.5), (2.7) and (2.8) hold in $R_{\delta_2,\eta}$. Fix $\gamma \in (0,\delta_2)$. For $(x,t) \in R^\gamma_{\delta_2,\eta}$, we have

$$\begin{aligned} L_3(\phi_3) \;=\; & \frac{\alpha}{(x-\zeta-\gamma/3)^2}\left\{\zeta' - \frac{2Mvv_x^{p-2}}{x-\zeta-\gamma/3} + A + B\right\} \\ & + \frac{\alpha}{(x-\zeta^*)^2}\left\{\zeta^{*\prime} - \frac{2Mvv_x^{p-2}}{x-\zeta^*} + A + B\right\} - C\phi_3 \\ & - D(\phi_3)^2 - Ev_x^{p-3}v_{xx}^3 - M(p-2)(p-3)(p-4)vv_x^{p-5}v_{xx}^4 \end{aligned}$$

where A, B, C, D, E and M are as before.

From (2.7), together with the fact that $x-\zeta^* \ge x-\zeta-\gamma/3$ we have

$$\begin{aligned} \frac{v}{x-\zeta^*} &\le \frac{v}{x-\zeta-\gamma/3} \le (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta}{x-\zeta-\gamma/3} \le (a+\epsilon)^{\frac{1}{p-1}}\frac{\gamma}{\gamma-\gamma/3} \\ &= \frac{3}{2}(a+\epsilon)^{\frac{1}{p-1}}. \end{aligned}$$

From (3.3), we have

$$D\alpha, D\beta < \frac{DK}{2} < DK \leq \frac{(p-1)a}{4} + \frac{(p-1)\epsilon}{4}. \tag{3.6}$$

Then since $|C|$ is bounded and from (2.5) and (2.7), we have

$$\begin{aligned} L_3(\phi_3) \quad & \geq \tfrac{\alpha}{Y^2}\left\{\tfrac{(p-1)a}{2} - \tfrac{3p+12M+1}{2}\epsilon - \delta_2(|C| - \overline{E}\tfrac{Y}{\alpha})\right\} \\ & + \tfrac{\beta}{(x-\zeta^*)^2}\left\{\tfrac{(p-1)a}{2} - \tfrac{3p+12M-1}{2}\epsilon - \delta_2(|C| - \overline{E}\tfrac{x-\zeta^*}{\beta})\right\} \end{aligned}$$

where $Y = x - \zeta - \gamma/3$ and $\overline{E} = |E| v_x^{p-3} v_{xx}^3$. Since ϵ satisfies (3.5) we can choose $\delta = \delta_2(\epsilon, a, m, p, C_2) > 0$ so small that $L_3(\phi_3) \geq 0$ in $R^{\gamma}_{\delta,\eta}$. □

Remark 3.1. *From (3.6) the Proposition 3.2 will be true for any* $\alpha, \beta \in (0, K)$.

Proposition 3.3. *(Barrier Transformation). Let* δ *and* η *be as in Proposition 3.2 with the additional restriction that*

$$\eta < \frac{\delta}{6\epsilon}, \tag{3.7}$$

where ϵ *is as in Proposition 3.2. Suppose that for some nonnegative constant* β

$$v^{(3)}(x,t) \leq \frac{\alpha}{x - \zeta(t)} + \frac{\beta}{x - \zeta^*(t)} \quad \text{in} \quad R_{\delta,\eta}. \tag{3.8}$$

Then $v^{(3)}$ *also satisfies*

$$v^{(3)}(x,t) \leq \frac{2\alpha/3}{x - \zeta(t)} + \frac{\beta + 2\alpha/3}{x - \zeta^*(t)} \quad \text{in} \quad R_{\delta,\eta}. \tag{3.9}$$

Proof. By Remark 3.1, for any $\gamma \in (0, \delta)$ since $\beta + 2\alpha/3 \leq K$ the function

$$\phi_3(x,t) = \frac{2\alpha/3}{x - \zeta - \gamma/3} + \frac{\beta + 2\alpha/3}{x - \zeta^*}$$

satisfies $L_3(\phi_3) \geq 0$ in $R^{\gamma}_{\delta,\eta}$. On the other hand, on the parabolic boundary of $R^{\gamma}_{\delta,\eta}$ we have $\phi_3 \geq v^{(3)}$. In fact, for $t = t_1$ and $\zeta_1 + \gamma \leq x \leq \zeta_1 + \delta$, with $\zeta_1 = \zeta(t_1)$, we have

$$\phi_3(x, t_1) = \frac{2\alpha}{x - \zeta_1 - \gamma/3} + \frac{\beta + 2\alpha/3}{x - \zeta_1} > \frac{4\alpha/3}{x - \zeta_1} + \frac{\beta}{x - \zeta_1} > v^{(3)}(x, t_1)$$

while for $x = \zeta + \delta$ and $t_1 \le t \le t_2$ we get, in view of (3.7),

$$\begin{aligned}\phi_3(\zeta+\delta,t) &\ge \frac{2\alpha/3}{\delta-\gamma/3} + \frac{\beta}{\zeta+\delta-\zeta^*} + \frac{2\alpha/3}{\delta+6\epsilon\eta} \\ &\ge \frac{2\alpha/3}{\delta} + \frac{\delta}{\zeta+\delta-\zeta^*} + \frac{\alpha/3}{\delta} \ge v^{(3)}(\zeta+\delta,t).\end{aligned}$$

Finally, for $x = \zeta + \gamma$, $t_1 \le t \le t_2$ we have

$$\phi_3(\zeta+\delta,t) = \frac{2\alpha/3}{\gamma-\gamma/3} + \frac{\beta+2\alpha/3}{\zeta+\gamma-\zeta^*} \ge \frac{\alpha}{\gamma} + \frac{\beta}{\zeta+\gamma-\zeta^*} \ge v^{(3)}(\zeta+\gamma,t).$$

By the comparison principle we get

$$\phi_3 \ge v^{(3)} \quad \text{in} \quad R^{\gamma}_{\delta.\eta}$$

for any $\gamma \in (0,\delta)$, and (3.9) follows by letting $\gamma \downarrow 0$. □

PROPOSITION 3.4. *Let $q = (x_0, t_0)$ be a point on the interface for which (2.1) holds. Then there exist constants C_3, δ and η depending only on p, q and u such that*

$$\left|\left(\frac{\partial}{\partial x}\right)^3 v\right| \le C_3 \quad \textit{in} \quad R_{\delta,\eta/2}.$$

Proof. By Proposition 3.1 we have, by letting $\alpha = 0$,

$$v^{(3)}(x,t) \le \frac{\beta}{x-\zeta^*} \le \frac{2\beta}{\epsilon\eta} \quad \text{in} \quad R_{\delta,\eta/2}.$$

Even though the equation (3.1) is not linear for $v^{(3)}$, a lower bound can be obtained in a similar way. □

4. Main Result

In this section we prove the interface is a C^∞ function in (t^*, ∞). First we find the estimates of the derivatives of the form

$$v^{(j)} \equiv \left(\frac{\partial}{\partial x}\right)^j v$$

for $j \geq 4$. For the porous medium equation, we have [2] the following equation:

$$\begin{aligned} L_j v^{(j)} &\equiv v_t^{(j)} - (m-1)vv_{xx}^{(j)} - (2+j(m-1))v_x v_x^{(j)} - c_{mj} v_{xx} v^{(j)} \\ &\quad - \sum_{l=3}^{j^*} d_{mj}^l v^{(l)} v^{(j+2-l)} = 0 \end{aligned}$$

for $j \geq 3$ in $P[u]$, where $j^* = [j/2]+1$, and the c_{mj} and d_{mj}^l are constants which depend only on their indices, but whose precise values are irrelevant. Note that L_j is linear in $v^{(j)}$. On the other hand for the p-Laplacian equation by a direct computation we have the following equation for $j \geq 4$,

$$\begin{aligned} L_j v^{(j)} &= v_t^{(j)} - Mvv_x^{p-2} v_{xx}^{(j)} - ((j-2)A+B)v_x^{(j)} - C_{pj} v^{(j)} \\ &\quad - F(v, v_x, \ldots, v^{(j-1)}) = 0 \end{aligned} \tag{4.1}$$

where A, B and M are as before, and C_{pj} involves only v and derivatives of order $< j$. Note that equation (4.1) is linear in $v^{(j)}$. We also follow the method in [2]. Hence our result is

PROPOSITION 4.1. *Let $q = (x_0, t_0)$ be a point on the interface for which (2.1) holds. For each integer $j \geq 2$ there exist constants C_j, δ and η depending only on j, m, p, q and u such that*

$$\left| \left(\frac{\partial}{\partial x} \right)^j v \right| \leq C_j \quad \textit{in} \quad R_{\delta, \eta/2}.$$

The proof also proceeds by induction on j. Suppose that $q = (x_0, t_0)$ is a point on the left interface for which (2.1) holds. Fix $\epsilon \in (0, a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_0 \equiv R_{\delta_0, \eta_0}(t_0) \subset P[u]$ and (2.6) holds. Thus we also have (2.7) and (2.8) in R_0. Assume that there are constants $C_k \in \mathbb{R}^+$ for $k = 3, \ldots, j-1$ such that

$$|v^{(k)}| \leq C_k \quad \text{on} \quad R_0 \quad for \quad k = 2, \ldots, j-1. \tag{4.2}$$

Observe that by Propositions 2.1, 2.2 and 3.4, (4.2) holds for $k = 2$ and $k = 3$.

By rescaling and interior estimates, we have

PROPOSITION 4.2. *There are constants $K \in \mathbb{R}^+$, $\delta \in (0, \delta_0)$, and $\eta \in (0, \eta_0)$ depending only on p,q and C_k for $k \in [2, j-1]$ with $j \geq 4$*

such that

$$|v^{(j)}(x,t)| \le \frac{K}{x-\zeta(t)} \quad in \quad R_{\delta,\eta}.$$

Proof. Set

$$\delta = \min\{\frac{2\delta_0}{3}, 2s\eta_0\},$$
$$\eta = \eta_0 - \frac{\delta}{4s},$$

and define

$$R(\overline{x},\overline{t}) \equiv \left\{(x,t) \in \mathbb{R}^2 : |x-\overline{x}| < \frac{\lambda}{2}, \overline{t} - \frac{\lambda}{4s} < t \le \overline{t}\right\}$$

for $(\overline{x},\overline{t}) \in R_{\delta,\eta}$, where $s = a+\epsilon$ and $\lambda = \overline{x} - \zeta(\overline{t})$. Then $(\overline{x},\overline{t}) \in R_{\delta,\eta}$ implies that $R(\overline{x},\overline{t}) \subset R_0$. Since $\delta_0 \ge \frac{3\delta}{2}$, $\lambda < \delta$ and ζ is nonincreasing, we have

$$t_0 - \eta_0 = t_0 - \eta - \frac{\lambda}{4s} < t < t_0 + \eta < t_0 + \eta_0$$

and

$$\overline{x} - \frac{\lambda}{2} = \overline{x} - \frac{\overline{x}+\zeta(\overline{t})}{2} = \frac{\overline{x}+\zeta(\overline{t})}{2} > \zeta(t_0+\eta_0)$$
$$\zeta(t_0-\eta) + \delta + \frac{\lambda}{2} < \zeta(t_0-\eta_0).$$

Also observe that for each $(\overline{x},\overline{t}) \in R_{\delta,\eta}$, $R(\overline{x},\overline{t})$ lies to the right of the line $x = \zeta(\overline{t}) + s(\overline{t}-t)$. Next set $x = \lambda\xi + \overline{x}$ and $t = \lambda\tau + \overline{t}$. The function

$$V^{(j-1)}(\xi,\tau) \equiv v^{(j-1)}(\lambda\xi+\overline{x}, \lambda\tau+\overline{t}) = v^{(j-1)}(x,t)$$

satisfies the equation

$$\begin{aligned} V_\tau^{(j-1)} \quad = \quad & \left\{M\frac{v}{\lambda}v_x^{p-2}V_\xi^{(j-1)} + [(j-2)A+B]v_x^{p-1}V^{(j-1)}\right\}_\xi \\ & -[Mv_x^{p-1} + M(p-2)vv_x^{p-3}v_{xx} + (j-2)A + B]V_\xi^{(j-1)} \qquad (4.3) \\ & +\lambda[C_{pj} - ((j-2)A_x + B_x)]V^{(j-1)} + \lambda F(v,\dots,v^{(j-2)}) \end{aligned}$$

in the region

$$B \equiv \left\{(\xi,\tau) \in \mathbb{R}^2 : |\xi| \le \frac{1}{2}, -\frac{1}{4s} < \tau \le 0\right\},$$

and $|V^{(j-1)}| \le C_{j-1}$ in B. In view of (2.7) and (2.8)

$$(a-\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda} \le \frac{v(x,t)}{\lambda} \le (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda}$$

and

$$\zeta(\bar{t}) \le \zeta(t) \le \zeta(\bar{t}) + s(\bar{t} - t) \le \zeta(\bar{t}) + \frac{\lambda}{4}.$$

Therefore

$$\frac{\lambda}{4} = \overline{x} - \frac{\lambda}{2} - \zeta(\bar{t}) - \frac{\lambda}{4} \le x - \zeta(t) \le \overline{x} + \frac{\lambda}{2} - \zeta(\bar{t}) = \frac{3\lambda}{2}$$

which implies

$$\frac{(a-\epsilon)^{\frac{1}{p-1}}}{4} \le \frac{v}{\lambda} \le \frac{3(a+\epsilon)^{\frac{1}{p-1}}}{2}.$$

Hence by (2.5) equation (3.2) is uniformly parabolic in B. Moreover, it follows from Propositions 2.1, 2.2 and 3.4 and by (4.2) that $V^{(j-1)}$ satisfies all of the hypotheses of Theorem 5.3.1 of [5]. Thus we conclude that there exists a constant $K = K(a, m, p, C_1, \ldots, C_{j-1}) > 0$ such that

$$\left| \frac{\partial}{\partial \xi} V^{(j-1)}(0,0) \right| \le K;$$

that is,

$$|v^{(j)}(\overline{x}, \bar{t})| \le \frac{K}{\lambda}.$$

Since $(\overline{x}, \bar{t}) \in R_{\delta,\eta}$ is arbitrary, this proves the proposition. □

We now turn to the barrier construction. If $\gamma \in (0, \delta)$ we will use the notation

$$R^{\gamma}_{\delta,\eta} = R^{\gamma}_{\delta,\eta}(t_0) \equiv \{(x,t) \in \mathbb{R}^2 : \zeta(t) + \gamma \le x \le \zeta(t) + \delta, t_0 - \eta \le t \le t_0 + \eta\}.$$

PROPOSITION 4.3. *Let R_{δ_1,η_1} be the region constructed in the proof of Proposition 2.2. For $j \ge 4$ and $(x,t) \in R^{\gamma}_{\delta_1,\eta_1}$, let*

$$\phi_j(x,t) \equiv \frac{\alpha}{x - \zeta(t) - \gamma/3} + \frac{\beta}{x - \zeta^*(t)} \tag{4.4}$$

where ζ^* is given by (2.9), and α and β are positive constant. Then there exist $\delta \in (0, \delta_1)$ and $\eta \in (0, \eta_1)$ depending only on a, p, $C_1, \ldots, C_{j-1}$ such that

$$L_j(\phi_j) \ge 0 \quad in \quad R^{\gamma}_{\delta,\eta}$$

for all $\gamma \in (0, \delta)$.

Proof. Choose ϵ such that

$$0 < \epsilon < \frac{(3M(j-3)+(j-2)p-1)a}{3M(j-1)+(j-2)p+2}. \tag{4.5}$$

There exist $\delta_2 \in (0, \delta_1)$ and $\eta \in (0, \eta_1)$ such that (2.5), (2.7) and (2.8) hold in $R_{\delta_2,\eta}$. Fix $\gamma \in (0, \delta_2)$. For $(x, t) \in R^\gamma_{\delta_2,\eta}$, we have

$$\begin{aligned} L_j(\phi_j) &= \frac{\alpha}{A^{*2}}\left\{\zeta' - \frac{2Mvv_x^{p-2}}{A^*} + (j-2)A + B - C_{pj}A^* + \frac{A^{*2}}{\alpha}F\right\} \\ &+ \frac{\beta}{(x-\zeta^*)^2}\left\{\zeta^{*'} - \frac{2Mvv_x^{p-2}}{x-\zeta^*} + (j-2)A + B - C_{pj}(x-\zeta^*)\right\} \end{aligned}$$

where A, B, M, C_{pj} and F are as before and $A^* = x - \zeta - \gamma/3$. From (2.7), together with the fact that $x - \zeta^* \geq x - \zeta - \gamma/3$ we have

$$\begin{aligned} \frac{v}{x-\zeta^*} &\leq \frac{v}{x-\zeta-\gamma/3} \leq (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta}{x-\zeta-\gamma/3} \leq (a+\epsilon)^{\frac{1}{p-1}}\frac{\gamma}{\gamma-\gamma/3} \\ &= \frac{3}{2}(a+\epsilon)^{\frac{1}{p-1}}. \end{aligned}$$

Then from (2.5), (2.7) and (4.2), we have

$$\begin{aligned} L_j(\phi_j) &\geq \frac{\alpha}{A^{*2}}\{(3M(j-3)+(j-2)p-1)a - (3M(j-1) \\ &+ (j-2)p+1)\epsilon - \delta_2(|C_{pj}| + \frac{\delta}{\alpha}|F|\} + \frac{\beta}{(x-\zeta^*)^2}\{(3M(j-3) \\ &+ (j-2)p-1)a - (3M(j-1)+(j-2)p+2)\epsilon - \delta_2(|C_{pj}|\}. \end{aligned}$$

Since ϵ satisfies (4.5) we can choose $\delta = \delta_2(\epsilon, a, m, p, C_2) > 0$ so small that $L_3(\phi_3) \geq 0$ in $R^\gamma_{\delta,\eta}$. □

Hence we have the following proposition whose proof can be found in [2].

Proposition 4.4. *(Barrier Transformation). Let δ and η be as in Proposition 4.3 with the additional restriction that*

$$\eta < \frac{\delta}{6\epsilon}, \tag{4.6}$$

where ϵ is as in Proposition 4.3. Suppose that for some nonnegative constant β

$$v^{(j)}(x,t) \leq \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad in \quad R_{\delta,\eta}. \tag{4.7}$$

Then $v^{(j)}$ *also satisfies*

$$v^{(j)}(x,t) \leq \frac{2\alpha/3}{x-\zeta(t)} + \frac{\beta+2\alpha/3}{x-\zeta^*(t)} \quad in \quad R_{\delta,\eta}. \tag{4.8}$$

Then as in [2], we can prove the C^∞ regularity of the interface.

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