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SOME PROPERTIES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

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Abstract. We investigate some properties of functions of $\Phi\Lambda$-bounded variation on a closed bounded interval, which is the generalization of bounded variation.

1. Introduction

The concept of bounded variation has been generalized in many ways. If f is a function of bounded variation, its Fourier series converges uniformly to f on a closed bounded set. The desire to extend this theorem to larger classes of functions has provided much of the impetus for the study of generalizations of bounded variation. Afterward two directions of research have been pursued. One is to find a function spaces where Fourier series converges pointwise to its associated function. The other is to find ways that modify its Fourier coefficients to make resulting series converge to associated function. This method is called summability. We can think a summability method on given $\Phi\Lambda BV$ space. This paper is aimed at studying $\phi\Lambda BV$.

2. The Relations Of Functions Of Generalized Bounded Variation

In defining a function of bounded variation on $[a, b]$, we considered the supremum of $\sum |f(I_n)|$ for every collection $\{I_n\}$ of non-overlapping subintervals of $[a, b]$ such that $[a, b] = \bigcup I_n$, where

$$I_n = [x_n, y_n], f(I_n) = f(x_n) - f(y_n).$$

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A function f is of bounded variation on $[a, b]$ if $V(f) = \sup \sum |f(I_n)|$ is finite, that is, if there exists a positive constant c such that for every collection $\{I_n\}$ of subintervals of $[a, b]$,

$$\sum |f(I_n)| \le c.$$

Schramm [3] generalized the above idea by considering a sequence of increasing convex functions $\phi = \{\phi_n\}$ defined on $[0, \infty)$; f is of ϕ-*bounded variation* on $[a, b]$ if $V_\phi f = \sup \sum \phi_n(|f(I_n)|)$ is finite. We denote by ϕBV the collection of all functions f such that cf is of ϕ-bounded variation for some $c > 0$.

If $\Lambda = \{\lambda_n\}$ is an increasing sequence of positive numbers such that $\sum \frac{1}{\lambda_n}$ diverges, the functions of Λ-*bounded variation* (ΛBV) are those f for which

$$\sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < \infty$$

for every sequence I_n of non-overlapping intervals.

Definition 2.1. If ϕ is a nonnegative convex function defined on $[0, \infty)$ such that $\frac{\phi(x)}{x} \to 0$ as $x \to 0$, we say that f is of $\phi\Lambda$-*bounded variation* if, for every $\{I_n\}$,

$$\sum \frac{\phi(x)(c|f(I_n)|)}{\lambda_n} < \infty$$

and the total variation of f over $[a, b]$ is defined by

$$V_{\phi\Lambda}(f) = \sup \sum \frac{\phi(|f(I_n)|)}{\lambda_n}.$$

We denote by $\phi\Lambda BV$ the collection of all functions f such that cf is of $\phi\Lambda$-bounded variation for some $c > 0$. We denote by $f \in \phi\Lambda BV$ if $V_{\phi\Lambda}(cf) < \infty$ for some $c > 0$.

Here, we investigate that if f is of ϕBV on $[a, b]$, then f is of $\phi\Lambda BV$ on $[a, b]$.

Proposition 2.2. *If f is of ϕBV on a closed interval $[a, b]$, then f is $\phi\Lambda BV$ on $[a, b]$.*

Proof. For every collection $\{I_n\}$ of non-overlapping subintervals of $[a, b]$ such that $[a, b] = \bigcup I_n$, we can make that

$$\sum_{n=1}^{N} \frac{\phi(|(cf)(I_n)|)}{\lambda_n} \le \sum_{n=1}^{N} \frac{\phi(|(cf)(I_n)|)}{\lambda_1} = \frac{1}{\lambda_1} \sum_{n=1}^{N} \phi(|(cf)(I_n)|) < \infty.$$

Thus, $V_{\phi\Lambda}(cf) < \infty$ and so $\phi BV \subset \phi\Lambda BV$. $\square$

3. $\phi\Lambda$-Bounded Variation

We will investigate that $\phi\Lambda BV_0$ is a Banach space and if f be of $\phi\Lambda$-bounded variation, then $V_{\phi\Lambda}$ is right(left) continuous at a point $x \in [a,b]$ if and only if f is right(left) continuous at x.

If ϕ is an increasing convex function, $\phi(0) = 0$, $x \geq 0$ and $0 \leq \alpha \leq 1$, we have $\phi(\alpha x) \leq \alpha\phi(x)$. Let $c_1 > 0$ be such that $V_\phi(c_1 f) < \infty$ and let $0 < c \leq c_1$.

Then

$$V_\phi(cf) \leq \frac{c}{c_1} V_\phi(c_1 f) \to 0$$

as $c \to 0$. Recall that $f \in \phi\Lambda BV$ if $V_{\phi\Lambda}(cf) < \infty$ for some $c > 0$. We define a norm as follows:

Let $\phi\Lambda BV_0 = \{f \in \phi\Lambda BV | f(a) = 0\}$. For $f \in \phi\Lambda BV_0$, let

$$\|f\| = \inf \big\{k > 0 \,|\, V_{\phi\Lambda}\left(\frac{f}{k}\right) \leq \frac{1}{\lambda_1}\big\}.$$

We show that $\|f\|$ is a norm in the following lemma.

Lemma 3.1. *$\|\cdot\|$ is a norm on $\phi\Lambda BV_0$.*

Proof. Since $\|f\| = \inf\{k > 0 \,|\, V_{\phi\Lambda}(\frac{f}{k}) \leq \frac{1}{\lambda_1}\}$, $\|f\| \geq 0$. If $f \neq 0$, let $x \in [a,b]$ be a point such that $f(x) \neq 0$. Then

$$V_{\phi\Lambda}(\frac{f}{k}) \geq \frac{\phi(|f(I_1)|/k)}{\lambda_1} \to \infty$$

as $k \to 0$. Thus there is a $k > 0$ so that $V_{\phi\Lambda}(\frac{f}{k}) > \frac{1}{\lambda_1}$, and so $\|f\| \neq 0$.

$$\begin{aligned}
\|cf\| &= \inf\big\{k>0 \,|\, V_{\phi\Lambda}(\frac{cf}{k}) \le \frac{1}{\lambda_1}\big\} = \inf\big\{k>0 | V_{\phi\Lambda}\big(\frac{|c|f}{k}\big) \le \frac{1}{\lambda_1}\big\}\\
&= |c|\,\|f\|.\\
&\sum_n \frac{\phi\big(\frac{|(f+g)(I_n)|}{\|f\|+\|g\|}\big)}{\lambda_n}\\
&\le \sum_n \big[\frac{\|f\|}{\|f\|+\|g\|}\cdot\frac{\phi(\frac{|(f)(I_n)|}{\|f\|})}{\lambda_n} + \frac{\|g\|}{\|f\|+\|g\|}\cdot\frac{\phi(\frac{|(g)(I_n)|}{\|g\|})}{\lambda_n}\big]\\
&\le \frac{\|f\|}{\|f\|+\|g\|}\cdot\frac{1}{\lambda_1} + \frac{\|g\|}{\|f\|+\|g\|}\cdot\frac{1}{\lambda_1} = \frac{1}{\lambda},
\end{aligned}$$

thus $\|f+g\| \le \|f\| + \|g\|$. □

LEMMA 3.2. (1) $V_{\phi\Lambda}(\frac{f}{\|f\|}) \le \frac{1}{\lambda_1}$.
(2) *if* $\|f\| \le 1$, *then* $V_{\phi\Lambda}(f) \le \frac{\|f\|}{\lambda_1}$.

Proof. (1) Take $k > \|f\|$; then for any finite collection $\{I_n\}$,

$$\sum_n \frac{\phi(|f(I_n)|/k)}{\lambda_n} \le V_{\phi\Lambda}\frac{f}{k} \le \frac{1}{\lambda_1}.$$

Thus

$$\begin{aligned}
\sum_n \frac{\phi(|f(I_n)|/\|f\|)}{\lambda_n} &= \lim_{k\to\|f\|+} \sum_n \frac{\phi(|f(I_n)|/k)}{\lambda_n}\\
&\le \lim_{k\to\|f\|+} \frac{1}{\lambda_1}\\
&= \frac{1}{\lambda_1},
\end{aligned}$$

which implies $V_{\phi\Lambda}(\frac{f}{\|f\|}) \le \frac{1}{\lambda_1}$.

(2) For any $\{I_n\}$, since $\|f\| \le 1$,

$$\sum_n \frac{\phi(|f(I_n)|)}{\lambda_n} \le \|f\| \sum_n \frac{\phi(|f(I_n)|/\|f\|)}{\lambda_n} \le \|f\|\cdot\frac{1}{\lambda_1} = \frac{\|f\|}{\lambda_1}.$$

Thus $V_{\phi\Lambda}(f) \le \frac{\|f\|}{\lambda_1}$. □

THEOREM 3.3. $(\phi\Lambda BV_0, \|\cdot\|)$ *is a Banach space.*

Proof. By (1) of Lemma 3.2, $(\phi\Lambda BV_0, \|\cdot\|)$ is a normed linear space with the norm $\|f\| = \inf\{k > 0 \ |V_{\phi\Lambda}(\frac{f}{k}) \le \frac{1}{\lambda_1}\}$. It is enough to show that $\phi\Lambda BV_0$ is complete.

Let f and g be functions in $\phi\Lambda BV_0$ such that $\|f-g\| < \varepsilon$. Then $\|\frac{f-g}{\varepsilon}\| < 1$, so, by lemma 3.2,

$$V_{\phi\Lambda}(\frac{f-g}{\varepsilon}) \le \frac{\|\frac{f-g}{\varepsilon}\|}{\lambda_1} < \frac{1}{\lambda_1}.$$

Now for $x \in [a,b]$,

$$\frac{\phi\frac{|f(x)-g(x)|}{\varepsilon}}{\lambda_1} < V_{\phi\Lambda}(\frac{f-g}{\varepsilon}) < \frac{1}{\lambda_1}.$$

This implies that if $\{f_n\}$ is a Cauchy sequence in this norm it is also a Cauchy sequence in the supremum norm. Thus there is a function f such that $f_n \to f$ uniformly. Let $\varepsilon > 0$ be given. Let $\{I_k\}$ be a finite collection of non-overlapping subintervals of $[a,b]$ such that $[a,b] = \bigcup I_k$, where

$$I_k = [x_k, y_k],\ f(I_k) = f(x_k) - f(y_k),$$

and suppose $\|f_n - f_m\| < \varepsilon$, then

$$\begin{aligned}\sum_k \frac{\phi(\frac{|(f_n-f)(I_k)|}{\varepsilon})}{\lambda_k} &= \lim_{m\to\infty} \sum_k \frac{\phi(\frac{|(f_n-f_m)(I_k)|}{\varepsilon})}{\lambda_k} \\ &\le \lim_{m\to\infty} V_{\phi\Lambda}(\frac{f_n-f_m}{\varepsilon}) \\ &\le \frac{1}{\lambda_1}.\end{aligned}$$

Thus $V_{\phi\Lambda}(\frac{f_n-f}{\varepsilon}) \le \frac{1}{\lambda_1}$, $f \in \phi\Lambda BV_0$ and $f_n \to f$ in norm. □

Lemma 3.4 (Waterman [6]). *Let f be of class ΛBV on $I = [a,b]$. If $[x,y] \subset I$ and $|f(x) - f(y)| \ge \delta > 0$, then $v(y) - v(x) \ge \frac{\delta}{\lambda_{k_0}}$ where*

$$k_0 = \inf\{k | \sum_{n=1}^{k} \frac{1}{\lambda_n} > \frac{2v(x)}{\delta}\}$$

and

$$v(x) = v_\Lambda(f,x) = V_\Lambda(f,[a,x]).$$

We shall show that v possesses a continuity property, and that the continuity properties of v are exactly those of the function from which it is derived.

LEMMA 3.5. *Let f be of class $\phi\Lambda BV$ on $I=[a,b]$. If $[x,y]\subset I$ and $|f(x)-f(y)|\geq\delta>0$, then $v(y)-v(x)\geq\frac{\delta}{2\lambda_{k_0}}$ where*

$$k_0=\inf\{k|\sum_{n=1}^{k}\frac{1}{\lambda_n}>\frac{2v(x)}{\delta}\}$$

and

$$v(x)=v_{\phi\Lambda}(f,x)=V_{\phi\Lambda}(f,[a,x]).$$

Proof. Given $\varepsilon>0$, there exist $I_n, n=1,\cdots,N$ in $[a,x]$, such that $\{|f(I_n)|\}$ is a decreasing sequence and

$$v(x)\leq\sum_{n=1}^{N}\frac{\phi(|f(I_n)|)}{\lambda_n}+\varepsilon.$$

Let $m=\inf(\{n:\phi(|f(I_n)|)<\frac{\delta}{2}\}\bigcup\{N+1\})$ and we claim that

$$v(y)-v(x)\geq\frac{\delta}{2\lambda_m}-\varepsilon. \tag{3.1}$$

Put $|f(I_n)|=a_n$, $T=\sum\limits_{n=1}^{N}\frac{\phi(a_n)}{\lambda_n}$.

If $\phi(a_n)\geq\frac{\delta}{2}$, $n=1,\cdots,k$, but $\phi(a_{k+1})<\frac{\delta}{2}$, set

$$S=\frac{\phi(a_1)}{\lambda_1}+\cdots+\frac{\phi(a_k)}{\lambda_k}+\frac{\delta}{\lambda_{k+1}}+\frac{\phi(a_{k+1})}{\lambda_{k+2}}+\cdots+\frac{\phi(a_N)}{\lambda_{N+1}};$$

then

$$\begin{aligned}S-T&=\frac{\delta-\phi(a_{k+1})}{\lambda_{k+1}}+\frac{\phi(a_{k+1})-\phi(a_{k+2})}{\lambda_{k+2}}+\cdots+\frac{\phi(a_N)}{\lambda_{N+1}}\\&>\frac{\delta-\phi(a_{k+1})}{\lambda_{k+1}}\geq\frac{\delta-\frac{\delta}{2}}{\lambda_{k+1}}=\frac{\delta}{2\lambda_{k+1}}.\end{aligned}$$

If $\phi(a_n)\geq\frac{\delta}{2}$ for all n, set

$$S=\frac{\phi(a_1)}{\lambda_1}+\cdots+\frac{\phi(a_N)}{\lambda_N}+\frac{\delta}{\lambda_{N+1}};$$

then

$$S - T = \frac{\delta}{\lambda_{N+1}} > \frac{\delta}{2\lambda_{N+1}};$$

If $\phi(a_n) < \frac{\delta}{2}$ for all n, set

$$S = \frac{\delta}{\lambda_1} + \frac{\phi(a_1)}{\lambda_2} + \cdots + \frac{\phi(a_N)}{\lambda_{N+1}};$$

then

$$\begin{aligned} S - T &= \frac{\delta - \phi(a_1)}{\lambda_1} + \frac{\phi(a_1) - \phi(a_2)}{\lambda_2} + \cdots + \frac{\phi(a_{N-1}) - \phi(a_N)}{\lambda_N} + \frac{\phi(a_N)}{\lambda_{N+1}} \\ &> \frac{\delta - \phi(a_1)}{\lambda_1} > \frac{\delta - \frac{\delta}{2}}{\lambda_1} = \frac{\delta}{2\lambda_1}. \end{aligned}$$

Now $v(y) \geq S$. Hence $v(y) - v(x) \geq v(y) - (T + \varepsilon) \geq S - T - \varepsilon$ and so

$$v(y) - v(x) \geq \begin{cases} \dfrac{\delta}{2\lambda_{k+1}} - \varepsilon & \text{if } \phi(a_n) \geq \frac{\delta}{2} \text{ for } n \leq k \text{ and } \phi(a_{k+1}) < \frac{\delta}{2} \\ \dfrac{\delta}{2\lambda_{N+1}} - \varepsilon & \text{if } \phi(a_n) \geq \frac{\delta}{2} \ \forall n \\ \dfrac{\delta}{2\lambda_1} - \varepsilon & \text{if } \phi(a_n) \geq \frac{\delta}{2} \ \forall n \end{cases}$$

which is (3.1).

Now $k_0 \geq m$ since

1) if $m = 1$, then $k_0 \geq 1$,

2) if $m = N + 1$, then

$$v(x) \geq \sum_{n=1}^{N} \frac{\phi(|f(I_n)|)}{\lambda_n} \geq \frac{\delta}{2} \sum_{n=1}^{N} \frac{1}{\lambda_n}$$

and so

$$k_0 \geq N + 1,$$

3) if $1 < m < N + 1$, then

$$v(x) \geq \sum_{n=1}^{N} \frac{\phi(|f(I_n)|}{\lambda_n} \geq \sum_{n=1}^{m-1} \frac{\phi(a_n)}{\lambda_n} > \frac{\delta}{2} \sum_{n=1}^{m-1} \frac{1}{\lambda_n}$$

and so

$$k_0 \geq m.$$

Since k_0 is independent to ε, the proof is complete. □

It is known that if f in ΛBV is right continuous on $[a, b]$, then V_Λ is right continuous on $[a, b]$ (cf. Waterman [6]).

THEOREM 3.6. *Let f be of $\phi\Lambda$-bounded variation. Then $V_{\phi\Lambda}$ is right (left) continuous at a point $x \in [a, b]$ if and only if f is right (left) continuous at x.*

Proof. We shall consider only the behavior at the right of a point. The arguments for the left are analogous. Suppose $I = [a, b]$, $a \le x < b$, and f is right continuous at x. If $x < y \le b$, then

$$0 \le V_{\phi\Lambda}([a, y]) - V_{\phi\Lambda}([a, x]) \le V_{\phi\Lambda}([x, y]).$$

Since f is right continuous at x,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } y - x < \delta \rightarrow |f(y) - f(x)| < \varepsilon.$$

Thus we have that

$$\phi(|f([x, y])|) = \phi(|f(y) - f(x)|) = \phi(0) = 0$$

and hence

$$V_{\phi\Lambda}([x, y]) = \sup \sum \frac{\phi(|f([x, y])|)}{\lambda_n} = 0$$

as $y \rightarrow x+$ and, therefore, $V_{\phi\Lambda}$ is continuous on the right at x.

Suppose f is not right continuous at x. Then there is a $\delta > 0$ such that for $y > x$ but sufficiently close to x, $|f(x) - f(y)| \ge \delta$. Applying Lemma 3.5, we see then that

$$V_{\phi\Lambda}([a, y]) - V_{\phi\Lambda}([a, x]) \ge \frac{\delta}{2\lambda_{k_0}}$$

for such y and, therefore, $V_{\phi\Lambda}$ is discontinuous on the right at x. □

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