STUDY ON BROWDER’S SPECTRUMS AND WEYL’S SPECTRUMS

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Abstract. In this paper we give several necessary and sufficient conditions for an operator on the Hilbert space \(H\) to obey Browder’s theorem. And it is shown that if \(S\) has totally finite ascent and \(T \prec S\) then \(f(T)\) obeys Browder’s theorem for every \(f \in H(\sigma(T))\), where \(H(\sigma(T))\) denotes the set of all analytic functions on an open neighborhood of \(\sigma(T)\).

Furthermore, it is shown that if \(T \in B(H)\) is a compact operator or a Riesz Operator then \(T\) obeys Browder’s theorem and Weyl’s theorem holds if and only if Browder’s holds.

1. Introduction

Throughout this note let \(B(H)\) and \(K(H)\) denote respectively the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional Hilbert space \(H\).

If \(T \in B(H)\) write \(N(T)\) and \(R(T)\) for the null space and the range of \(T\); \(\alpha(T) = \dim N(T)\); \(\beta(T) = \text{codim} R(T)\); \(\sigma(T)\) for the spectrum of \(T\); \(\pi_0(T)\) for the set of eigenvalues of \(T\); \(\pi_{0f}(T)\) for the eigenvalues of finite multiplicity; \(\pi_{0lf}(T)\) for the isolated points of \(\sigma(T)\); which are eigenvalues of finite multiplicity; \(\pi_{00}(T) = \sigma(T) \setminus \sigma_b(T)\) for the Riesz Points of \(T\).

An operator \(T \in B(H)\) is called Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension.

The index of a Fredholm Operator \(T \in B(H)\) is given by \(i(T) = \alpha(T) - \beta(T)\).

An operator \(T \in B(H)\) is called Weyl if it is Fredholm of index zero. An operator \(T \in B(H)\) is called Browder if it is Fredholm “of finite
ascent and descent”; equivalently ([9, Theorem 7.9.3]) if \( T \) is Fredholm and \( T - \lambda I \) is invertible for sufficiently small \( \lambda \neq 0 \) in \( \mathbb{C} \).

The essential spectrum \( \sigma_e(T) \), the Weyl’s spectrum \( w(T) \), the Browder’s spectrum \( \sigma_b(T) \), the regular spectrum \( \sigma_r(T) \) of \( T \in B(H) \) are defined by ([8, 9]):

\[
\sigma_e(T) = \{ \lambda \in \mathbb{C}; T - \lambda I \text{ is not Fredholm } \}; \\
w(T) = \{ \lambda \in \mathbb{C}; T - \lambda I \text{ is not Weyl } \}; \\
\sigma_b(T) = \{ \lambda \in \mathbb{C}; T - \lambda I \text{ is not Browder } \}; \\
\sigma_r(T) = \{ \lambda \in \mathbb{C}; T - \lambda I \text{ is not regular } \};
\]

By [9, Theorem 6.4.2] \( \sigma_r(T) \subset \sigma_e(T) \).

Evidently \( \sigma_r(T) \subseteq \sigma_e(T) \subseteq w(T) \subseteq (\sigma_b(T) \cap \sigma_e(T)) \cup acc\sigma(T) \) where we write \( accK \) for the accumulation points of \( K \subseteq \mathbb{C} \).

We say Weyl’s theorem holds for \( T \in B(H) \) if there is equality (1.1)\( \sigma(T) \setminus w(T) = \pi_0^{left}(T) \) and that Browder’s theorem holds for \( T \in B(H) \) if there is equality (1.2)\( \sigma(T) \setminus \sigma_b(T) = \pi_00(T) \).

An operator \( T = B(H) \) is a \( \text{Gm} \)-operator \( (m \geq 1) \) if there exists a constant \( M \) such that \( \| (T - \lambda I) \| \leq \frac{M}{(d(\lambda, \sigma(T)))^m} \) for every \( \lambda \notin \sigma(T) \).

The condition \( N\lambda \) is said to be satisfied at a particular \( \lambda N(T - \lambda I) \cap N[(T - \lambda I)^*]^n \) is nontrivial for some positive integer \( n \), which may depend on \( \lambda \). An operator \( T \in B(H) \) is said to be dominant if for every \( \lambda \in \mathbb{C} \) there exists a constant \( M\lambda \) such that \( (T - \lambda I)(T - \lambda I)^* \leq M\lambda(T - \lambda I)^*(T - \lambda I) \) and an operator \( T \in B(H) \) is said to be paranormal if \( \|Tx\|^2 \leq ||T^2x)||x|| \) for all \( x \in H \).

In particular, \( T \) is called totally paranormal if \( T - \lambda I \) is paranormal for every \( \lambda \in \mathbb{C} \).

\( T \in B(H) \) is called a quasi-affinity if it has trivial kernel and dense range.

\( S \in B(H) \) is said to be a quasi-affine transform of \( T \in B(H) \) (notation; \( S \prec T \)) if there is a quasi-affinity \( K \in B(H) \) such that \( KS = TK \).

If both \( S \prec T \) and \( T \prec S \), then we say that \( S \) and \( T \) are quasisimilar.

An operator \( T \in B(H) \) has totally finite ascent if \( T - \lambda I \) has finite ascent for each \( \lambda \in \mathbb{C} \).

It is known ([10]) that if \( T \in B(H) \) then we have; Weyl’s theorem \( \Rightarrow \) Browder’s theorem.

If \( T \in B(H) \) is a compact operator then it is proved that Weyl’s theorem holds if and only if Browder’s theorem holds where \( T \in B(H) \).
is a compact operator on $H$ if for any bounded sequence $< x_n >_{n=1}^{\infty}$ in $H$, the sequence $< Tx_n >_{n=1}^{\infty}$ has a convergent subsequence $< T_{x_{n_k}} >_{k=1}^{\infty}$ such that this limit $\lim_{k \to \infty} T_{x_{n_k}} = T_{x_0} \in H$.

Example: Let $l_2 = \{ < x_n >_{n=1}^{\infty} : \text{sequences in } \mathbb{C}, | \sum_{n=1}^{\infty} |x_n|^2 < +\infty \}$ be a Hilbert space.

Consider $T = B(l_2)$ defined by $T(< x_n >_{n=1}^{\infty}) = < \frac{x_n}{n} >_{n=1}^{\infty}$. Then by [18, Theorem 2], $T$ is a compact operator on $l_2$. Since $Tx_n = \frac{x_n}{n}$, we have $\sigma(T) = \{ \frac{1}{n} | n \in \mathbb{N} \} \cup \{0\}$.

2. Main result

**Theorem 2.1.** Let $T \in B(H)$. Then the following statements are equivalent:

1. $T$ obeys Browder’s theorem;
2. $\sigma(T) \setminus w(T) \subseteq \text{iso}\sigma(T)$;
3. $\gamma T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T) \setminus w(T)$, where $\gamma T(\cdot)$ denotes the reduced minimum modulus;
4. Every $\lambda \in \sigma(T) \setminus w(T)$ satisfies the condition $N\lambda$;
5. $T - \lambda I$ has finite ascent for each $\lambda \in \sigma(T) \setminus w(T)$.

**Proof.** (1)$\iff$ (2): If $T$ obeys Browder’s theorem then $\sigma(T) \setminus w(T) = \pi_{00}(T) \subseteq \text{iso}\sigma(T)$. Conversely, suppose that $\lambda \in \sigma(T) \setminus w(T)$. Then $T - \lambda I$ is Weyl. But $\lambda \in \text{iso}\sigma(T)$; hence by the punctured neighborhood theorem $\lambda \in \sigma(T) \setminus \sigma_0(T)$. Therefore $T$ obeys Browder’s theorem.

(1)$\iff$(3): If $T$ obeys Browder’s theorem then it follows from [6, Lemma 5.52] that $\gamma T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T) \setminus w(T)$. Conversely, suppose that $\gamma T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T) \setminus w(T)$. Let $\lambda_0 \in \sigma(T) \setminus w(T)$. Then $T - \lambda_0 I$ is Weyl and $\alpha(T - \lambda_0 I) > 0$. Therefore $\gamma T(\lambda) > 0$ for all $\lambda$ near $\lambda_0$, and so by [6, Corollary 5.74] $\alpha(T - \lambda I) < \alpha(T - \lambda_0 I)$; for otherwise $\gamma T(\lambda)$ would be continuous at $\lambda_0$. Since all nearby values $\lambda$ are also in $\sigma(T) \setminus w(T)$, the discontinuity of $\gamma T(\lambda)$ requires that $\alpha(T - \lambda I) = 0$ in $\sigma(T) \setminus w(T)$. Therefore $\lambda_0$ is an isolated point of $\sigma(T)$.

(1)$\iff$(4): The forward implication follows from [7, Theorem 1]. Conversely, suppose that $\lambda_0 \in \sigma(T) \setminus w(T)$. Then $T - \lambda_0 I$ is Weyl and $\alpha(T - \lambda_0 I) > 0$. Since every $\lambda \in \sigma(T) \setminus w(T)$ satisfies the condition
We claim that \( N_{\lambda} \), by the punctured neighborhood theorem there exists a neighborhood \( N(\lambda_0; p) \) for some \( p > 0 \) such that \( \alpha(T - \lambda I) \) is constant (say no) on \( N(\lambda_0; p) \setminus \{ \lambda_0 \} \) and \( 0 \leq \alpha(T - \lambda I) < \alpha(T - \lambda_0) \). We now claim that \( n_0 = 0 \). Assume to the contrary that \( n_0 \neq 0 \). Also by the punctured neighborhood theorem there exists a neighborhood \( N(\lambda_0; q) \) for some \( q > 0 \) such that \( \lambda_1 \in N(\lambda_0; q) \setminus \{ \lambda_0 \} \) implies \( \alpha(T - \lambda_1 I) > 0 \) and \( T - \lambda_1 I \) is Weyl. Thus we have \( \lambda_1 \in \sigma(T) \setminus w(T) \). Now by the same reason as for \( \lambda_0 \); there exists a neighborhood \( N(\lambda_1; r) \) for \( r > 0 \) such that \( \alpha(T - \mu I) \) is constant (say \( n_1 \)) and \( 0 \leq \alpha(T - \mu I) < \alpha(T - \lambda_1 I) \). Thus \( \lambda \in [N(\lambda_0; q) \cap N(\lambda_1, r)] \setminus \{ \lambda_0, \lambda_1 \} \Rightarrow \alpha(T - \lambda I) = n_1 < n_0 \), a construction. Therefore \( n_0 = 0 \) and hence \( \lambda_0 \) is an isolated point of \( \sigma(T) \). Hence it follows from (2) that Browder’s theorem holds for \( T \).

(1) \( \Leftrightarrow \) (5): If \( T \) obeys Browder’s theorem then \( \sigma(T) \setminus w(T) = \pi_{00}(T) \). Therefore \( T - \lambda I \) has finite ascent for each \( \lambda \in \sigma(T) \setminus w(T) \). Conversely, Suppose that \( T - \lambda I \) has finite ascent for each \( \lambda \in \sigma(T) \setminus w(T) \). Then by the Index Product Theorem, \( \alpha((T - \lambda I)^p) - \beta(T - \lambda I)^n = i((T - \lambda I)^n) - n \cdot i(T - \lambda I) = 0 \). Thus if \( \alpha((T - \lambda I)^p) \) is a constant then so is \( \beta((T - \lambda I)^n) \). Therefore \( T - \lambda I \) is Weyl. Thus \( T \) obeys Browder’s theorem.

We can’t expect that Weyl’s theorem holds for operators having totally finite ascent. Consider the following example:

Let \( T \in B(l_2) \) be defined by \( T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \ldots) \). Then \( T \) has totally finite ascent. But \( \sigma(T) = w(T) = \{ 0 \} \) and \( \pi_{00}^T(T) = \phi; \) hence Weyl’s theorem does not hold for \( T \). However, Browder’s theorem performs better:

**Corollary 2.2.** Suppose that \( S \in B(H) \) has totally finite ascent and \( T \in B(H) \) satisfies \( T \vartriangleleft S \). Then \( f(T) \) obeys Browder’s theorem for every \( f \in H(\sigma(T)) \). In particular, if \( S \) is a dominant operator and \( T \vartriangleleft S \) then Browder’s theorem holds for \( f(T) \) for every \( f \in H(\sigma(T)) \), where \( H(\sigma(T)) \) denotes the set of all analytic functions on an open neighborhood of \( \sigma(T) \).

**Proof.** Since \( T \vartriangleleft S \), then exists a quasiaffinity \( K \in B(H) \) such that \( KT = SK \). But \( S \) has totally finite ascent; hence for each \( \lambda \) there exists a natural number \( n_\lambda \) such that \( N((S - \lambda I)^{n_\lambda}) = N((S - \lambda I)^{n_\lambda + 1}) \).

We claim that \( N((T - \lambda I)^{n_\lambda}) = N((T - \lambda I)^{n_\lambda + 1}) \). Let \( x \in N((TS -
\begin{align*}
&\text{conclusion is evident from the precious assertion.} \\
&\text{for all} \ T \ \text{from [10, Theorem 14] that} \\
&\text{finite ascent then} \\
&\text{spectral subspace} \\
&\text{Then by Corollary 2.2} \\
&\text{has an analytic solution} \\
&\text{Recall that if} \\
&\lambda \ \text{ascent then} \\
&\text{commute,} \\
&\text{where} \\
&\text{Browder and hence} \\
&\text{right side of (2.2.1) commute,} \\
&\text{is a dominant operator, then} \\
&\text{is isoloid then} \\
&\text{is Weyl and (2.2.1):} \\
&\text{if} \ A \ \text{has finite descent then} \ A \ \text{is Browder and hence} \ i(A) = 0, \ \text{or if} \ A \ \text{does not have finite descent} \\
&\text{as} \ n \ \to \ -\infty \ \text{as} \ n \ \to \ \infty, \ \text{which implies that} \\
&\text{i(A) < 0. Therefore} \ \lambda \neq w(f(T)), \ \text{and hence} \ f(w(T)) = w(f(T)). \ \text{Hence} \\
&\text{so Browder’s theorem holds for} \ f(T). \ \text{If} \ S \ \text{is a dominant operator, then} \ N(S - \lambda I) \subseteq N(S^* - \lambda I) \ \text{for all} \ \lambda \in \mathbb{C}. \ \text{Therefore} \ S \ \text{has totally finite ascent, and hence the} \\
&\text{conclusion is evident from the precious assertion.} 
\end{align*}

\textbf{Corollary 2.3.} \textit{Let} \ T \in B(H) \ \text{be a} \ \textbf{Gm}-\text{operator. If} \ T \ \text{has totally finite ascent then} \ f(T) \ \text{obeys Weyl’s theorem for every} \ f \in H(\sigma(T)). \ \textbf{Proof.} \text{Since} \ T \ \text{has totally finite ascent, it follows from Theorem 2.1 that} \ T \ \text{obeys Browder’s theorem. But} \ T \ \text{is a} \ \textbf{Gm}-\text{operator, it follows from [10, Theorem 14] that} \ T \ \text{obeys Weyl’s theorem. Let} \ f \in H(\sigma(T)). \ \text{Then by Corollary 2.2} \ f(w(T)) = w(f(T)). \ \text{Remembering ([13, Lemma]) that if} \ T \ \text{is isoloid then} \\
&f(\sigma(T) \setminus \pi_{0}^{lef}(T)) = \sigma(f(T) \setminus \pi_{0}^{lef}(T)) \ \text{for every} \ f \in H(\sigma(T)). \ \text{Hence} \ \sigma(f(T)) \setminus \pi_{0}^{lef}(f(T)) = f(\sigma(T) \setminus \pi_{0}^{lef}(T)) - f(w(T)) = w(f(T)), \ \text{which implies that Weyl’s theorem holds for} \ f(T).} 

\text{Recall that if} \ T \in B(H) \ \text{and} \ F \ \text{is a closed subset of} \ \mathbb{C} \ \text{then we define a} \\
\text{spectral subspace} \ \mathcal{H}T(F) \ \text{as follows;} \ \mathcal{H}T(F) = \{x \in H : (T - \lambda I)f(\lambda) = x \ \text{has an analytic solution} \ f : \mathbb{C} \setminus F \to H\}. 

\textbf{Theorem 2.4.} \textit{Let} \ T \in B(H). \ \text{If} \ \mathcal{H}T(\{\lambda\}) = N(T - \lambda I) \ \text{for every} \ \lambda \in \pi_{0f}(T), \ \text{then} \ T \ \text{obeys Weyl’s theorem.}
Proof. Let \( \lambda \in \sigma(T) \setminus w(T) \). Then \( \lambda \in \pi_0 f(T) \), and so \( \mathcal{H}T(\lambda) = N(T - \lambda I) \). Since \( \mathcal{H}T(\{\lambda\}) \) is invariant under \( T \), \( T \) can be represented as the following 2 \times 2 operator matrix with respect to the decomposition \( \mathcal{H}T(\{\lambda\}) = \mathcal{H}T(\{\lambda\}) \oplus \mathcal{H}T(\{\lambda\}) \):

\[
T = \begin{pmatrix} \lambda & T_1 \\ 0 & T_2 \end{pmatrix}
\]

Therefore \( \lambda \in \text{iso}\sigma(T) \), and hence \( \lambda \in \pi_0^{lef\, f}(T) \). Conversely, let \( \lambda \in \pi_0^{lef\, f}(T) \). Then using the spectral projection \( P = \frac{1}{2\pi i} \int_{\partial D} (\lambda I - T)^{-1}d\lambda \) where \( D \) is an open disk of center \( \lambda \) which contains no other points of \( \sigma(T) \), we can represent \( T \) as the direct sum \( T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \), where \( \sigma(T_1) = \{\lambda\} \) and \( \sigma(T_2) \setminus \{\lambda\} \). Since \( P(H) = \{x \in H : \lim ||(T - \lambda I)x||^{\frac{1}{n}} = 0\} = \mathcal{H}T(\{\lambda\}) \) and \( \mathcal{H}T(\{\lambda\}) \) is finite dimensional, \( w(T) = w(T_2) \). But \( (T - \lambda I) \) is invertible; hence \( T - \lambda I \) Weyl. Therefore \( \lambda \in \sigma(T) \setminus w(T) \). \( \square \)

**Corollary 2.5.** If \( T \in B(H) \) is totally paranormal operator then Weyl’s theorem holds for \( f(T) \) for every \( f \in H(\sigma(T)) \).

**Proof.** If \( T \) is totally paranormal, then it follows [11, Corollary 4.8] that \( \mathcal{H}T(\{\lambda\}) = N(T - \lambda I) \) for every \( \lambda \in \mathbb{C} \). Therefore by Theorem 2.4 Weyl’s theorem holds for \( T \). But \( T \) has totally finite ascent and \( T \) is an isloid, it follows from the proof of Corollary 2.3 that \( f(T) \) obeys Weyl’s theorem. \( \square \)

**Theorem 2.6.** Let \( T \in B(H) \) be a compact operator. Then

(1) Browder’s theorem holds for \( T \);

(2) Weyl’s theorem holds for \( T \) if and only if Browder’s theorem holds for \( T \);

(3) The spectrum of \( T \) \( \sigma(T) \) is a compact set.

**Proof.** Let \( B(H)/K(H) \) be a Calkin algebra and let \( \pi : B(H) \rightarrow B(H)/K(H) \) be the natural map. Then by [19, Definition 2.1] \( \sigma_\pi(T) = \sigma(\pi(T)) \). And so if \( T \) is compact then \( \sigma_\pi(T) = \sigma(\pi(T)) = \{0\} \).

(1) Since \( T \) is compact; \( \sigma_\pi(T) = \{0\} \) and \( \sigma_\pi(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_\pi(T) \cup \text{acc}\sigma(T) \), we have \( \sigma(T) = \{0\} \) and so \( w(T) = \{0\} \). Thus \( \text{acc}\sigma(T) \subseteq w(T) \) by [10, Theorem 9] Browder’s theorem holds.
(2) Assume that Weyl’s theorem holds. Then \( \sigma(T) \setminus w(T) = \pi^{lef}(T) \). But \( T \) is compact, \( w(T) = \{0\} = \sigma_b(T) \). And so \( \sigma(T) \setminus \sigma_b(T) = \pi_0(T) \). Since Browder’s theorem holds for \( T \), by [10, Theorem 2] \( \sigma(T) = w(T) \cup \pi^{lef}(T) \). And so \( \sigma(T) \setminus w(T) = \pi^{lef}(T) \).

(3) Since \( T \) is a compact operator on \( H \), \( \sigma(T) \) is both closed and bounded. And so \( \sigma(T) \) is a compact set. \( \square \)

**Definition 2.7.** Let \( T \in B(H) \). \( T \) is said be a Riesz Operator if it has the following three properties:

1. For every \( \lambda \neq 0 \) and each positive integer \( n \), the set of solutions of the equations \( (\lambda I - T)^n(x) = 0 \) forms a finite-dimensional subspace of \( X \), which is independent of \( n \) provided that \( n \) is sufficiently large.
2. For every \( \lambda \neq 0 \) and each positive integer \( n \), the range of \( (\lambda I - T)^n \) is a closed subspace of \( X \) which is independent of \( n \) provided that \( n \) is sufficiently large.
3. The eigenvalues of \( T \) have at most one cluster point 0.

**Corollary 2.8.** Let \( T \in B(H) \) be a Riesz Operator. Then

1. Browder’s theorem holds for \( T \);
2. Weyl’s theorem holds for \( T \) if and only if Browder’s theorem holds for \( T \).

**Proof.** By [16, Theorem 3.14] \( \sigma(T) \) is countable and has no cluster point except possibly 0. And so Corollary 2.8 is easily proved from Theorem 2.6 \( \square \).

Furthermore, if the spectral radius \( \nu(T) \) of \( T \in B(H) \) is zero number, that is; \( \nu(T) = 0 \), then by [16, Proposition 1.10] Weyl’s theorem and Browder’s theorem hold for \( T \).

**References**


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