Kangweon-Kyungki Math. Jour. **12** (2004), No. 2, pp. 147–154

STUDY ON BROWDER'S SPECTRUMS AND WEYL'S SPECTRUMS

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Abstract. In this paper we give several necessary and sufficient conditions for an operator on the Hilbert space H to obey Browder's theorem. And it is shown that if S has totally finite ascent and $T \prec S$ then $f(T)$ obeys Browder's theorem for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

Furthermore, it is shown that if $T \in B(H)$ is a compact operator or a Riesz Operator then T obeys Browder's theorem and Weyl's theorem holds if and only if Browder's holds.

1. Introduction

Throughout this note let $B(H)$ and $K(H)$ denote respectively the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional Hilbert space H.

If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and the range of T;$\alpha(T) = \dim N(T)$;$\beta(T) =$codim$R(T)$;$\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T;$\pi_{0f}(T)$ for the eigenvalues of finite multiplicity;$\pi_0^{left}(T)$ for the isolated points of $\sigma(T)$; which are eigenvalues of finite multiplicity; $\pi_{00}(T) = \sigma(T)\backslash\sigma_b(T)$ for the Riesz Points of T.

An operator $T \in B(H)$ is called Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension.

The index of a Fredholm Operator $T \in B(H)$ is given by $i(T) = \alpha(T) - \beta(T)$.

An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. An operator $T \in B(H)$ is called Browder if it is Fredholm "of finite

Received May 27, 2004. Revised August 22, 2004.
2000 Mathematics Subject Classification: 47A53 .
Key words and phrases: Browder's spectrums, Weyl's spectrums.
This study was supported by 2002 Kangwon National University Research Fund.

ascent and descent": equivalently ([9, Theorem 7.9.3]) if T is Fredholm and $T-\lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$.

The essential spectrum $\sigma_e(T)$, the Weyl's spectrum $w(T)$, the Browder's spectrum $\sigma_b(T)$, the regular spectrum $\sigma_r(T)$ of $T \in B(H)$ are defined by ([8, 9]):

$$\begin{aligned}
\sigma_e(T) &= \{\lambda \in \mathbb{C}; T-\lambda I \text{ is not Fredholm }\};\\
w(T) &= \{\lambda \in \mathbb{C}; T-\lambda I \text{ is not Weyl }\};\\
\sigma_b(T) &= \{\lambda \in \mathbb{C}; T-\lambda I \text{ is not Browder }\};\\
\sigma_r(T) &= \{\lambda \in \mathbb{C}; T-\lambda I \text{ is not regular }\};
\end{aligned}$$

By [9, Theorem 6.4.2] $\sigma_r(T) \subset \sigma_e(T)$.

Evidently $\sigma_r(T) \subseteq \sigma_e(T) \subseteq w(T) \subseteq (\sigma_b(T) \cap \sigma_e(T)) \cup acc\sigma(T)$ where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.

We say Weyl's theorem holds for $T \in B(H)$ if there is equality $(1.1)\sigma(T)\backslash w(T) = \pi_0^{left}(T)$ and that Browder's theorem holds for $T \in B(H)$ if there is equality $(1.2)\sigma(T)\backslash\sigma_b(T) = \pi_{00}(T)$.

An operator $T = B(H)$ is a **Gm**-operator ($m \geq 1$) if there exists a constant M such that $||(T-\lambda I)|| \leq \frac{M}{(d(\lambda \cdot \sigma(T)))^m}$ for every $\lambda \notin \sigma(T)$.

The condition $N\lambda$ is said to be satisfied at a particular $\lambda N(T-\lambda I) \cap N[((T-\lambda I)^*)^n]$ is nontrivial for some positive integer n, which may depend on λ. An operator $T \in B(H)$ is said to be dominant if for every $\lambda \in \mathbb{C}$ there exists a constant $M\lambda$ such that $(T-\lambda I)(T-\lambda I)^* \leq M\lambda(T-\lambda I)^*(T-\lambda I)$ and an operator $T \in B(H)$ is said to be paranormal if $||Tx||^2 \leq ||T^2x||||x||$ for all $x \in H$

In Particular, T is called totally paranormal if $T-\lambda I$ is paranormal for every $\lambda \in \mathbb{C}$.

$T \in B(H)$ is called a quasiaffinity if it has trivial kernel and dense range.

$S \in B(H)$ is said to be a quasiaffine transform of $T \in B(H)$ (notation; $S \prec T$) if there is a quasiaffinity $K \in B(H)$ such that $KS = TK$.

If both $S \prec T$ and $T \prec S$; then we say that S and T are quasisimilar.

An operator $T \in B(H)$ has totally finite ascent if $T-\lambda I$ has finite ascent for each $\lambda \in \mathbb{C}$.

It is known ([10]) that if $T \in B(H)$ then we have; Weyl's theorem $\Rightarrow$ Browder's theorem.

If $T \in B(H)$ is a compact operator then it is proved that Weyl's theorem holds if and only if Browder's theorem holds where $T \in B(H)$

is a compact operator on H if for any bounded sequence $< x_n >_{n=1}^{\infty}$ in H, the sequence $< Tx_n >_{n=1}^{\infty}$ has a convergent subsequence $< T_{x_{n_k}} >_{k=1}^{\infty}$ such that this limit $\lim_{k\to\infty} T_{x_{n_k}} = T_{x_0} \in H$.

Example: Let $l_2 = \{< x_n >_{n=1}^{\infty}$: sequences in $\mathbb{C}, |\sum_{n=1}^{\infty} |x_n|^2 < +\infty\}$ be a Hilbert space.

Consider $T = B(l_2)$ defined by $T(< x_n >_{n=1}^{\infty}) =< \frac{x_n}{n} >_{n=1}^{\infty}$. Then by [18, Theorem 2], T is a compact operator on l_2. Since $Tx_n = \frac{x_n}{n}$, we have $\sigma(T) = \{\frac{1}{n}|n \in N\} \cup \{0\}$.

2. Main result

THEOREM 2.1. *Let $T \in B(H)$. Then the following statements are equivalent:*

(1) *T obeys Browder's theorem;*
(2) *$\sigma(T)\backslash w(T) \subseteq iso\sigma(T)$;*
(3) *$\gamma T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T)\backslash w(T)$, where $\gamma T(\cdot)$ denotes the reduced minimum modulus;*
(4) *Every $\lambda \in \sigma(T)\backslash w(T)$ satisfies the condition $N\lambda$;*
(5) *$T - \lambda I$ has finite ascent for each $\lambda \in \sigma(T)\backslash w(T)$.*

Proof. (1)$\Leftrightarrow$ (2) :If T obeys Browder's theorem then $\sigma(T)\backslash w(T) = \pi_{00}(T) \subseteq iso\sigma(T)$. Conversely, suppose that $\lambda \in \sigma(T)\backslash w(T)$. Then $T - \lambda I$ is Weyl. But $\lambda \in iso\sigma(T)$; hence by the punctured neighborhood theorem $\lambda \in \sigma(T)\backslash\sigma_b(T)$. Therefore T obeys Browder's theorem.

(1)$\Leftrightarrow$(3): If T obeys Browder's theorem then it follows from [6, Lemma 5.52] that $\gamma T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T)\backslash w(T)$. Conversely, suppose that $\gamma T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T)\backslash w(T)$. Let $\lambda_0 \in \sigma(T)\backslash w(T)$. Then $T - \lambda_0 I$ is Weyl and $\alpha(T - \lambda_0 I) > 0$. Therefore $\gamma T(\lambda) > 0$ for all λ near λ_0, and so by [6, Corollary 5.74] $\alpha(T - \lambda I) < \alpha(T - \lambda_0 I)$; for otherwise $\gamma T(\lambda)$ would be continuous at λ_0. Since all nearby values λ are also in $\sigma(T)\backslash w(T)$, the discontinuity of $\gamma T(\lambda)$ requires that $\alpha(T - \lambda I) = 0$ in $\sigma(T)\backslash w(T)$. Therefore λ_0 is an isolated point of $\sigma(T)$.

(1)$\Leftrightarrow$(4): The forward implication follows from [7, Theorem 1]. Conversely, suppose that $\lambda_0 \in \sigma(T)\backslash w(T)$. Then $T - \lambda_0 I$ is Weyl and $\alpha(T - \lambda_0 I) > 0$. Since every $\lambda \in \sigma(T)\backslash w(T)$ satisfies the condition

N_λ, by the punctured neighborhood theorem there exists a neighborhood $N(\lambda_0; P)$ for some $P > 0$ such that $\alpha(T - \lambda I)$ is constant (say no) on $N(\lambda_0; P)\backslash\{\lambda_0\}$ and $0 \leq \alpha(T - \lambda I) < \alpha(T - \lambda_0)$. We now claim that $n_0 = 0$. Assume to the contrary that $n_0 \neq 0$. Also by the punctured neighborhood theorem there exists a neighborhood $N(\lambda_0; q)$ for some $q > 0$ such that $\lambda_1 \in N(\lambda_0; q)\{\lambda_0\}$ implies $\alpha(T - \lambda_1 I) > 0$ and $T - \lambda_1 I$ is Weyl. Thus we have $\lambda_1 \in \sigma(T)\backslash w(T)$. Now by the same reason as for λ_0; there exists a neighborhood $N(\lambda_1; r)$ for $r > 0$ such that $\alpha(T - \mu I)$ is constant (say n_1) and $0 \leq \alpha(T - \mu I) < \alpha(T - \lambda_1 I)$. Thus $\lambda \in [N(\lambda_0; q) \cap N(\lambda_1, r)]\backslash\{\lambda_0, \lambda_1\} \Leftrightarrow \alpha(T - \lambda I) = n_1 < n_0$, a construction. Therefore $n_0 = 0$ and hence λ_0 is an isolated point of $\sigma(T)$. Hence it follows from (2) that Browder's theorem holds for T.

(1)$\Leftrightarrow$(5): If T obeys Browder's theorem then $\sigma(T)\backslash w(T) = \pi_{00}(T)$. Therefore $T - \lambda I$ has finite ascent for each $\lambda \in \sigma(T)\backslash w(T)$. Conversely, Suppose that $T - \lambda I$ has finite ascent for each $\lambda \in \sigma(T)\backslash w(T)$. Then by the Index Product Theorem, $\alpha((T - \lambda I)^n) - \beta(T - \lambda I)^n = i((T - \lambda I)^n) - n \cdot i(T - \lambda I) = 0$. Thus if $\alpha((T - \lambda I)^n)$ is a constant then so is $\beta((T - \lambda I)^n)$. Therefore $T - \lambda I$ is Browder. Thus T obeys Browder's theorem. □

We can't expect that Weyl's theorem holds for operators having totally finite ascent. Consider the following example:

Let $T \in B(l_2)$ be defined by $T(x_1, x_2, x_3, \dots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$. Then T has totally finite ascent. But $\sigma(T) = w(T) = \{0\}$ and $\pi_0^{left}(T) = \phi$; hence Weyl's theorem does not hold for T. However, Browder's theorem performs better:

COROLLARY 2.2. *Suppose that $S \in B(H)$ has totally finite ascent and $T \in B(H)$ satisfies $T \prec S$. Then $f(T)$ obeys Browder's theorem for every $f \in H(\sigma(T))$. In particular, if S is a dominant operator and $T \prec S$ then Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.*

Proof. Since $T \prec S$, then exists a quasiaffinity $K \in B(H)$ such that $KT = SK$. But S has totally finite ascent; hence for each λ there exists a natural number $n\lambda$ such that $N((S - \lambda I)^{n\lambda}) = N((S - \lambda I)^{n\lambda+1})$. We claim that $N((T - \lambda I)^{n\lambda}) = N((T - \lambda I)^{n\lambda+1})$. Let $x \in N((TS -$

$\lambda I)^{n\lambda})$, Then $(T-\lambda I)^{n\lambda+1}x=0$, and so $(S-\lambda I)^{n\lambda+1}Kx=K(T-\lambda I)^{n\lambda+1}x=0$. Therefore $Kx\in N((S-\lambda I)^{n\lambda+1})=N((S-\lambda I)^{n\lambda})$, and so $(S-\lambda I)^{n\lambda}Kx=0$. Since $K(T-\lambda I)^{n\lambda}x=0$ and K is a quasiaffinity, $x\in N(T-\lambda I)^{n\lambda}$. Since T has totally finite ascent, it follows from Theorem 2.1 that $w(T)=\sigma_b(T)$. Let $f\in H(\sigma(T))$. We shall show that $w(f(T))\subseteq f(w(T))$ for every $f\in H(\sigma(T))$ with no other restriction on T ([5, Theorem 2]), it suffices to show that $f(w(T))\subset w(f(T))$.

Suppose that $\lambda\notin w(f(T))$. Then $f(T)-\lambda I$ is Weyl and (2.2.1):

$f(T)-\lambda I=c(T-\alpha_1 I)(T-\alpha_2 I)\cdots(T-\alpha_n I)g(T)$,

where $c,\alpha_1,\alpha_2,\cdots,\alpha_n\in\mathbb{C}$ and $g(T)$ is invertible. Since the operators right side of (2.2.1) commute, $(T-\alpha_i I)$ is Fredholm. Now we show that $i(T-\alpha_i I)\le 0$. Observe that if $A\in B(H)$ is Fredholm of finite ascent then $i(A)\le 0$: Indeed, either if A has finite descent then A is Browder and hence $i(A)=0$, or if A does not have finite descent then $n\cdot i(A)-\alpha(A^n)-\beta(A^n)\to-\infty$ as $n\to\infty$, which implies that $i(A)<0$. Therefore $\lambda\notin f(w(T))$, and hence $f(w(T))=w(f(T))$. Hence $\sigma_b(f(T))=f(\sigma_b(T))=f(w(T))=w(f(T))$,and so Browder's theorem holds for $f(T)$. If S is a dominant operator, then $N(S-\lambda I)\subseteq N(S^*-\overline{\lambda I})$ for all $\lambda\in\mathbb{C}$. Therefore S has totally finite ascent, and hence the conclusion is evident from the precious assertion. □

Corollary 2.3. *Let $T\in B(H)$ be a **Gm**-operator. If T has totally finite ascent then $f(T)$ obeys Weyl's theorem for every $f\in H(\sigma(T))$.*

Proof. Since T has totally finite ascent, it follows from Theorem 2.1 that T obeys Browder's theorem. But T is a **Gm**-operator, it follows from [10, Theorem 14] that T obeys Weyl's theorem. Let $f\in H(\sigma(T))$. Then by Corollary 2.2 $f(w(T))=w(f(T))$. Remembering ([13, Lemma]) that if T is isoloid then $f(\sigma(T)\backslash\pi_0^{left}(T))=\sigma(f(T)\backslash\pi_0^{left})$ for every $f\in H(\sigma(T))$. Hence $\sigma(f(T))\backslash\pi_0^{left}(f(T))=f(\sigma(T)\backslash\pi_0^{left}(T))-f(w(T))=w(f(T))$, which implies that Weyl's theorem holds for $f(T)$. □

Recall that if $T\in B(H)$ and F is a closed subset of $\mathbb{C}$ then we define a spectral subspace $\mathcal{H}T(F)$ as follows; $\mathcal{H}T(F)=\{x\in H:(T-\lambda I)f(\lambda)=x$ has an analytic solution $f:\mathbb{C}\backslash F\to H\}$.

Theorem 2.4. *Let $T\in B(H)$. If $\mathcal{H}T(\{\lambda\})=N(T-\lambda I)$ for every $\lambda\in\pi_{0f}(T)$, then T obeys Weyl's theorem.*

Proof. Let $\lambda \in \sigma(T)\backslash w(T)$. Then $\lambda \in \pi_{0f}(T)$, and so $\mathcal{H}T(\lambda) = N(T-\lambda I)$. Since $\mathcal{H}T(\{\lambda\}$ is invariant under T, T can be represented as the following 2×2 operator matrix with respect to the decomposition $\mathcal{H}T(\{\lambda\}) \oplus \mathcal{H}T(\{\lambda\})^{\perp}$:
$T = \begin{pmatrix} \lambda & T_1 \\ 0 & T_2 \end{pmatrix}$ and $\mathcal{H}T(\{\lambda\})$ is finite dimension, $T_2 - \lambda I$ is invertible. Therefore $\lambda \in iso\sigma(T)$, and hence $\lambda \in \pi_0^{left}(T)$. Conversely, let $\lambda \in \pi_0^{left}(T)$. Then using the spectral projection $P = \frac{1}{2\pi i} \int\limits_{\partial D} (\lambda I - T)^{-1} d\lambda$ where D is an open disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$,where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2)\backslash\{\lambda\}$. Since $P(H) = \{x \in H : \lim ||(T-\lambda I)_x^n||^{\frac{1}{n}} = 0\} = \mathcal{H}T(\{\lambda\})$ and $\mathcal{H}T(\{\lambda\})$ is finite dimensional, $w(T) = w(T_2)$. But $(T - \lambda I)$ is invertible; hence $T - \lambda I$ Weyl. Therefore $\lambda \in \sigma(T)\backslash(T)$. $\square$

COROLLARY 2.5. *If $T \in B(H)$ is totally paranormal operator then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. If T is totally paranormal, then it follows [11, Corrollary 4.8] that $\mathcal{H}T(\{\lambda\}) = N(T - \lambda I)$ for every $\lambda \in \mathbb{C}$. Therefore by Theorem 2.4 Weyl's theorem holds for T. But T has totally finite ascent and T is an isoloid, it follows from the proof of Corollary 2.3 that $f(T)$ obeys Weyl's theorem $\square$

THEOREM 2.6. *Let $T \in B(H)$ be a compact operator. Then*

(1) *Browder's theorem holds for T;*
(2) *Weyl's theorem holds for T if and only if Browder's theorem holds for T;*
(3) *The spectrum of T $\sigma(T)$ is a compact set.*

Proof. Let $B(H)/K(H)$ be a Calkin algebra and let $\pi : B(H) \to B(H)/K(H)$ be the natural map. Then by [19, Definition 2.1] $\sigma_e(T) = \sigma(\pi(T))$. And so if T is compact then $\sigma_e(T) = \sigma(\pi(T)) = \{0\}$.

(1) Since T is compact; $\sigma_e(T) = \{0\}$; $acc\sigma T = \{0\}$ and $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup acc\sigma(T)$, we have $\sigma(T) = \{0\}$ and so $w(T) = \{0\}$. Thus $acc\sigma(T) \subseteq w(T)$ by [10, Theorem 9] Browder's theorem holds.

(2) Assume that Weyl's theorem holds. Then $\sigma(T)\backslash w(T) = \pi_0^{left}(T)$. But T is compact, $w(T) = \{0\} = \sigma_b(T)$. And so $\sigma(T)\backslash w(T) = \sigma(T)\backslash\sigma_b(T)$. Assume that Browder's theorem holds. Then $\sigma(T)\backslash\sigma_b(T) = \pi_{00}(T)$. Since Browder's theorem holds for T, by [10, Theorem 2] $\sigma(T) = w(T)\cup \pi_0^{left}(T)$. And so $\sigma(T)\backslash w(T) = \pi_0^{left}(T)$.

(3) Since T is a compact operator on H, $\sigma(T)$ is both closed and bounded. And so $\sigma(T)$ is a compact set. □

Definition 2.7. *Let $T \in B(H)$. T is said be a Riesz Operator if it has the following three properties:*

(1) *For every $\lambda \neq 0$ and each positive integer n, the set of solutions of the equations $(\lambda I - T)^n(x) = 0$ forms a finite-dimensional subspace of X, which is independent of n provided that n is sufficiently large.*
(2) *For every $\lambda \neq 0$ and each positive integer n, the range of $(\lambda I - T)^n$ is a closed subspace of X which is independent of n provided that n is sufficiently large.*
(3) *The eigenvalues of T have at most one cluster point 0.*

Corollary 2.8. *Let $T \in B(H)$ be a Riesz Operator. Then*

(1) *Browder's theorem holds for T;*
(2) *Weyl's theorem holds for T if and only if Browder's theorem holds for T.*

Proof. By [16, Theorem 3.14] $\sigma(T)$ is countable and has no cluster point except possibly 0. And so Corollary 2.8 is easily proved from Theorem 2.6 □

Furthermore, if the spectral radius $\nu(T)$ of $T \in B(H)$ is zero number, that is ; $\nu(T) = 0$, then by [16, Proposition 1.10] Weyl's theorem and Browder's theorem hold for T.

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