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HALF-GP-MAPS ON INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

Won Keun Min, Chun-Kee Park and Kyung-Ho Min

Abstract. In this paper, we introduce the concepts of half-interior, half-closure, half-gp-maps and half-gp-open maps defined by intuitionistic gradations of openness.

1. Introduction

Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of fuzzy set in Zadeh's sense [8]. Çoker introduced the concept of intuitionistic fuzzy topological spaces [4], which it is an extended concept of fuzzy topological spaces [2] in Chang's sense. In [6], Mondal and Samanta introduced and investigated the concept of intuitionistic gradation of openness which is a generalization of the concept of gradation of openness defined by Chattopadyay [3, 5].

In this paper, we introduce the concepts of half-interior, half-closure, half-gp-map and half-gp-open map and also obtain some characterizations.

2. Preliminaries

Let X be a set and $I = [0, 1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

Definition 2.1 ([3,5, 7]). Let X be a non-empty set and $\tau : I^X \to I$ be a mapping satisfying the following conditions:
(O1) $\tau(0_X) = \tau(1_X) = 1$;

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(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(O3) For every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$.

Then the mapping $\tau : I^X \to I$ is called a *fuzzy topology* (or *gradation of openness* [7]) on X. We call the ordered pair (X, τ) a *fuzzy topological space*. The value $\tau(A)$ is called the *degree of openness* of A.

Definition 2.2 ([1]). An *intuitionistic fuzzy set* A in a set X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Definition 2.3 ([6]). An *intuitionistic gradation of openness* (briefly IGO) of fuzzy subsets of a set X is an ordered pair (τ, τ^*) of functions $\tau, \tau^* : I^X \to I$ such that
(IGO1) $\tau(A) + \tau^*(A) \leq 1$, for all $A \in I^X$;
(IGO2) $\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0$;
(IGO3) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B)$;
(IGO4) For every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$ and $\tau^*(\cup_{i\in J} A_i) \leq \vee_{i\in J} \tau^*(A_i)$.

Then the triplet (X, τ, τ^*) is called an *intuitionistic fuzzy topological space* (briefly $IFTS$) on X. τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.4 ([6]). Let X be a nonempty set and two functions $\mathcal{F}, \mathcal{F}^* : I^X \to I$ be satisfying
(IGC1) $\mathcal{F}(A) + \mathcal{F}^*(A) \leq 1$, for all $A \in I^X$;
(IGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = 1, \mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = 0$;
(IGC3) $\forall A, B \in I^X,\ \mathcal{F}(A \cup B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$ and $\mathcal{F}^*(A \cup B) \leq \mathcal{F}^*(A) \vee \mathcal{F}^*(B)$;
(IGC4) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \mathcal{F}(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \mathcal{F}(A_i)$ and $\mathcal{F}^*(\cap_{i\in J} A_i) \leq \vee_{i\in J} \mathcal{F}^*(A_i)$.

Then the ordered pair $(\mathcal{F}, \mathcal{F}^*)$ is called an *intuitionistic gradation of closedness* [6] (briefly IGC) on X. $\mathcal{F}$ and $\mathcal{F}^*$ may be interpreted as gradation of closedness and gradation of nonclosedness, respectively.

Theorem 2.5 ([6]). *Let X be a nonempty set. If (τ, τ^*) is an IGO on X, then the pair $(\mathcal{F}, \mathcal{F}^*)$, defined by $\mathcal{F}_\tau(A) = \tau(A^c)$, $\mathcal{F^*}_{\tau^*}(A) = \tau^*(A^c)$ where A^c denotes the complement of A, is an IGC on X. And if $(\mathcal{F}, \mathcal{F}^*)$ is an IGC on X, then the pair $(\tau_\mathcal{F}, \tau^*{}_{\mathcal{F}^*})$, defined by $\tau_\mathcal{F}(A) = \mathcal{F}(A^c)$, $\tau^*{}_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c)$ is an IGO on X.*

Definition 2.6 ([6]). Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a *gp-map* if $\tau(f^{-1}(A)) \geq \sigma(A)$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$ for every $A \in I^Y$.

3. Half-Closure and Half-Interior Operators in IFTS

In this section, we introduce the concepts of half-closure and half-interior of a fuzzy set on IFTS and investigate some their properties.

Definition 3.1. Let (X, τ, τ^*) be an IFTS and $A \in I^X$. Then the *half-closure* (resp., *half-interior*) of A, denoted by A_- (resp., A_o), is defined by $A_- = \cap\{K \in I^X : \mathcal{F}_\tau(A) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(A) \leq \frac{1}{2}, A \subseteq K\}$ (resp., $A_o = \cup\{K \in I^X : \tau(K) > 0 \text{ and } \tau^*(A) \leq \frac{1}{2}, K \subseteq A\}$).

Theorem 3.2. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$. Then*

1. $A_o \subseteq A \subseteq A_-$,
2. *If $A \subseteq B$, then $A_- \subseteq B_-$ and $A_o \subseteq B_o$,*
3. $(A_o)^c = A^c{}_-$,
4. $(A_-)^c = (A^c)_o$.

Proof. (1) and (2) follow directly from Definition 3.1.
(3) We have that

$$\begin{aligned}(A_-)^c &= (\cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \leq \frac{1}{2}, A \subseteq K\})^c \\ &= \cup\{K^c : K \in I^X, \tau(K^c) > 0 \text{ and } \tau^*(K^c) \leq \frac{1}{2}, K^c \subseteq A^c\} \\ &= \cup\{U \in I^X : \tau(U) > 0 \text{ and } \tau^*(U) \leq \frac{1}{2}, U \subseteq A^c\} \\ &= (A^c)_o.\end{aligned}$$

(4) It is similar to (3). □

Theorem 3.3. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$. Then*

1. $(0_X)_- = 0_X$,

2. $A \subseteq A_-$,
3. $A_- = (A_-)_-$,
4. $A_- \cup B_- \subseteq (A \cup B)_-$.

Proof. (1) and (2) follow from Definition 3.1.
(3) We have that

$$\begin{aligned}(A_-)_- &= \cap\{H \in I^X : \mathcal{F}_\tau(H) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(H) \leq \frac{1}{2}, A_- \subseteq H\} \\ &= \cap\{H \in I^X : \mathcal{F}_\tau(H) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(H) \leq \frac{1}{2}, H \supseteq \cap\{U \in I^X : \\ &\mathcal{F}_\tau(U) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(U) \leq \frac{1}{2}, A \subseteq U\}\} \\ &\subseteq \cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \leq \frac{1}{2}, A \subseteq K\} \\ &= A_-.\end{aligned}$$

Thus we get $(A_-)_- = A_-$ from Theorem 3.2.
(4) It follows from Theorem 3.2. □

THEOREM 3.4. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$. Then*
1. $(1_X)_o = 1_X$;
2. $A_o \subseteq A$;
3. $(A_o)_o = A_o$;
4. $(A \cap B)_o \subseteq A_o \cap B_o$.

Proof. The proof is similar to the proof of Theorem 3.3. □

THEOREM 3.5. *Let (X, τ, τ^*) be an IFTS and $A \in I^X$. Then*
1. $\tau(A) > 0$ *and* $\tau^*(A) \leq \frac{1}{2} \Rightarrow A_o = A$,
2. $\mathcal{F}_\tau(A) > 0$ *and* $\mathcal{F}^*_{\tau^*}(A) \leq \frac{1}{2} \Rightarrow A_- = A$.

Proof. (1) Let $\tau(A) > 0$ and $\tau^*(A) \leq \frac{1}{2}$. Then $A \in \{K \in I^X : \tau(K) > 0 \text{ and } \tau^*(A) \leq \frac{1}{2}, K \subseteq A\}$. From Definition 3.1 and Theorem 3.2, it follows $A_o = A$.

(2) It is similar to (1). □

EXAMPLE 3.6. Let $X = I$ and let N denote the set of all natural numbers. For each $n \in N$, we define a fuzzy set μ_n in X as the following: $\mu_n(x) = \frac{n-1}{n}x$ for $x \in X$.

Define an intuitionistic gradation of openness $\tau, \tau^* : I^X \to I$ by

$$\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0,$$
$$\tau(\mu_n) = \frac{1}{n}, \tau^*(\mu_n) = \frac{n-1}{2n},$$
$$\tau(\mu) = 0, \tau^*(\mu) = \frac{1}{2} \text{ for all other fuzzy set } \mu \in I^X.$$

Take a fuzzy set A in X such that $A(x) = x$ for all $x \in X$. Then it follows $A_o = A$ but $\tau(A) = 0,\ \tau^*(A) = \frac{1}{2}$. Thus the converse of the part (1) in Theorem 3.5 is not true in general.

In the same way, we can show that the converses of another implication in Theorem 3.5 is not true.

4. Half-gp-maps and Half-gp-open maps

Definition 4.1. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a *half-gp-map* iff for every $A \in I^Y$ such that $\sigma(A) > 0$ and $\sigma^*(A) \le \frac{1}{2}$, $\tau(f^{-1}(A)) > 0$ and $\tau^*(f^{-1}(A)) \le \frac{1}{2}$.

We recall that for two IFTSs (X, τ, τ^*) and (Y, σ, σ^*), a mapping $f : X \to Y$ is a *gp-map* if $\tau(f^{-1}(A)) \ge \sigma(A)$ and $\tau^*(f^{-1}(A)) \le \sigma^*(A)$ for every $A \in I^Y$.

Thus every gp-map is also a half-gp-map, but the converse may not be true as the following example.

Example 4.2. Let $X = I$ and lgt N denote the set of all natural numbers. For each $n \in N$, we consider $\mu_n \in I^X$ such that $\mu_n(x) = \frac{1}{n}x$ for $x \in X$.

Define $\tau, \tau^* : I^X \to I$ by

$$\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0;$$
$$\tau(\mu_n) = \frac{1}{n+2}, \tau^*(\mu_n) = \frac{1}{n+2} \text{ for each } n \in N;$$
$$\tau(\mu) = 0, \tau^*(\mu) = 1 \text{ for all other fuzzy set } \mu \in I^X.$$

And define $\sigma, \sigma^* : I^X \to I$ by

$$\sigma(0_X) = \sigma(1_X) = 1, \sigma^*(0_X) = \sigma^*(1_X) = 0;$$
$$\sigma(\mu_n) = \frac{1}{n+1}, \sigma^*(\mu_n) = \frac{1}{n+1} \text{ for each n} \in N;$$
$$\sigma(\mu) = 0, \sigma^*(\mu_n) = 1 \text{ for all other fuzzy set } \mu \in I^X.$$

Then the pairs (τ, τ^*) and (σ, σ^*) are two intuitionistc gradations of openness on X.

Consider the identity mapping $f : (X, \tau, \tau^*) \to (Y, \sigma, \sigma^*)$. Then f is a half-gp-map but not a gp-map because for each $n \in N$, $\tau^*(f^{-1}(\mu_n)) \leq \sigma^*(\mu_n)$ but $\sigma(\mu_n)$ is not less than $\tau(\mu_n)$.

THEOREM 4.3. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a half-gp-map iff for every $A \in I^Y$ such that $\mathcal{F}_\sigma(A) > 0$ and $\mathcal{F}^*_{\sigma^*}(A) \leq \frac{1}{2}$, $\mathcal{F}_\tau(f^{-1}(A)) > 0$ and $\mathcal{F}^*_{\tau^*}(f^{-1}(A)) \leq \frac{1}{2}$.*

Proof. It follows from Theorem 2.5 □

THEOREM 4.4. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. If $f : X \to Y$ is a half-gp-map, then we have*

1. *$f(A_-) \subseteq (f(A))_-$ for every $A \in I^X$,*
2. *$(f^{-1}(A))_- \subseteq f^{-1}(A_-)$ for every $A \in I^Y$,*
3. *$f^{-1}(A_o) \subseteq (f^{-1}(A))_o$ for every $A \in I^Y$.*

Proof. (1) For every $A \in I^X$, by Theorem 4.3 we have

$$\begin{aligned} f^{-1}(f(A)_-) &= f^{-1}(\cap\{U \in I^Y : \mathcal{F}_\sigma(U) > 0 \text{ and } \mathcal{F}^*{}_{\sigma^*}(U) \leq \frac{1}{2}, f(A) \subseteq U\}) \\ &\supseteq \cap\{f^{-1}(U) \in I^X : \mathcal{F}_\sigma(U) > 0 \text{ and } \mathcal{F}^*{}_{\sigma^*}(U) \leq \frac{1}{2}, A \subseteq f^{-1}(U)\} \\ &= \cap\{f^{-1}(U) \in I^X : \mathcal{F}_\tau(f^{-1}(U)) > 0 \text{ and } \mathcal{F}^*{}_{\tau^*}(f^{-1}(U)) \leq \frac{1}{2}, A \subseteq f^{-1}(U)\} \\ &\supseteq A_-. \end{aligned}$$

(2) It follows from (1).

(3) It is obtained by (2) and Theorem 3.2. □

DEFINITION 4.5. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is called a *half-gp-open* map if for every $A \in I^X$ $\tau(A) > 0$ and $\tau^*(A) \leq \frac{1}{2}$, then $\sigma(f(A)) > 0$ and $\sigma^*(f(A)) \leq \frac{1}{2}$.

THEOREM 4.6. *Let* (X, τ, τ^*) *and* (Y, σ, σ^*) *be two IFTSs. If* $f : X \to Y$ *is a half-gp-open map, then* $f(A_o) \subseteq (f(A))_o$ *for every* $A \in I^X$.

Proof. For every $A \in I^X$, we have

$$\begin{aligned} f(A_o) &= f(\cup\{U \in I^X : \tau(U) > 0 \text{ and } \tau^*(U) \leq \frac{1}{2}, U \subseteq A\}) \\ &\subseteq \cup\{f(U) \in I^Y : \tau(U) > 0 \text{ and } \tau^*(U) \leq \frac{1}{2}, f(U) \subseteq f(A)\} \\ &\subseteq \cup\{f(U) \in I^Y : \sigma(f(U)) > 0 \text{ and } \sigma^*(f(U)) \leq \frac{1}{2}, f(U) \subseteq f(A)\} \\ &\subseteq \cup\{K \in I^Y : \sigma(K) > 0 \text{ and } \sigma^*(K) \leq \frac{1}{2}, K \subseteq f(A)\} \\ &= (f(A))_o. \end{aligned}$$

□

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Department of Mathematics
Kangwon National University
Chuncheon, 200-701, Korea
E-mail: `wkmin@kangwon.ac.kr, ckpark@kangwon.ac.kr`