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AN APPLICATION OF p-ADIC ANALYSIS TO WINDOWED FOURIER TRANSFORM

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Abstract. We shall introduce the notion of the windowed Fourier transform in $\mathbb{Q}_p$ and show that, for any given function $g \in L^2(\mathbb{Q}_p)$ of norm, the windowed Fourier transform of f with respect to g be a function of norms, and moreover be expressible to a summation form. The results obtained in this paper will be usable to the field of research in data compression for signal processing according to the following scheme.

1. Introduction

The field $\mathbb{Q}_p$ of the p-adic numbers is defined as the completion of the field $\mathbb{Q}$ of rationals with respect to the p-adic metric induced by the p-adic norm $|\cdot|_p$. Nonzero p-adic number x can uniquely expressed by the canonical form

$$x = p^{-\gamma} \sum_{k=0}^{\infty} x_k p^k, \qquad |x|_p = p^{\gamma}, \tag{1.1}$$

where $\gamma \in \mathbb{Z}$ and $x_k \in \mathbb{Z}$ such that $0 \leq x_k \leq p-1, x_0 \neq 0$.

Fourier Analysis has been used as an analysis tool for the signal processing. In the p-adic analysis, Fourier transform of $f \in L^2(\mathbb{Q}_p)$, defined by

$$F(\xi) = \int_{\mathbb{Q}_p} f(x)\chi_p(\xi x)dx, \quad \chi_p(\xi x) \stackrel{\text{def}}{=} \exp(2\pi i\{\xi x\}_p),$$

may be one of the most important parts in the field of application. Here we note that if $f \in L^2(\mathbb{Q}_p)$ is a function of norm $|x|_p$, then its Fourier transform $F(\xi)$ is a function of norm $|\xi|_p$.

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Throughout the present paper we shall deal with a complex valued function of $p-$adic variable and we shall also call it a step function if it has finite range on each circle $|x|_p = const$ in $\mathbb{Q}_p$. We shall introduce the notion of the windowed Fourier transform in $\mathbb{Q}_p$ and show that, for any given function $g \in L^2(\mathbb{Q}_p)$ of norm, the windowed Fourier transform of f with respect to g be a function of norms, and moreover be expressible to a summation form. The results obtained in this paper will be usable to the field of research in data compression for signal processing according to the following scheme. Let a signal $f(t)$ be given, where t denotes the times variable.

(1) A positive real number $x \in \mathbb{R}^+$ can be uniquely expressed by the form

$$x = p^{\gamma} \sum_{k=0}^{\infty} x_k p^{-k},$$

where $\gamma \in \mathbb{Z}$ and $0 \le x_k \le p-1$, $x_0 \neq 0$ provided that we exclude the case that all except finitely many x_k and $p-1$. Hence we can introduce a mapping $P : \mathbb{R}^+ \to \mathbb{Q}_p$ by

$$P(0) = 0, \qquad P(p^{\gamma} \sum_{k=0}^{\infty} x_k p^{-k}) = p^{-\gamma} \sum_{k=0}^{\infty} x_k p^{k}.$$

The mapping P is clearly 1-1 but not onto. Hence there exists the left inverse $P_* : \mathbb{Q}_p \to \mathbb{R}^+$ of P such that

$$P_*(0) = 0, \qquad P_*(p^{-\gamma} \sum_{k=0}^{\infty} x_k p^{k}) = p^{\gamma} \sum_{k=0}^{\infty} x_k p^{-k}.$$

It is notable that the range of P is countable set consisting of the $p-$adic numbers of the form

$$x = p^{-\gamma}(\sum_{k=0}^{\infty} x_k p^k + (p-1) \sum_{k=n+1}^{\infty} p^k), \quad x_n \neq p-1,$$

for some integer $n \ge 0$, and that the range of P is dense in $\mathbb{Q}_p$.

(2) For a given signal $f(t)$, $t \in \mathbb{R}^+$, we consider a function $f_p : \mathbb{Q}_p \to \mathbb{R}$ defined by $f_p = f \circ P_*$.

(3) We could obtain much information about f_p for the data compression by using the Fourier transform in $\mathbb{Q}_p$ and then transmit and receive it, and do inverse Fourier transform of it in $\mathbb{Q}_p$. Finally we could obtain desirable information about the original signal f by virtue of $f = f_p \circ P$.

2. Main theorems

PROPOSITION 1. *Let $f \in L^2(\mathbb{Q}_p)$ be a function of norm $|x|_p$. Then, for each $\psi \in L^2(\mathbb{Q}_p)$, the following equality is valid :*

$$\int_{\mathbb{Q}_p} f(|x|_p)\psi(\xi x)dx = \sum_{\gamma\in\mathbb{Z}} f(p^\gamma)\int_{S_\gamma} \psi(|\xi|_p^{-1}x)dx, \quad (\xi \in \mathbb{Q}_p) \tag{2.1}$$

Proof. Let $\xi = |\xi|_p^{-1}(\xi_0 + \xi_1 p + + \cdots)$ and $\eta = (\xi_0 + \xi_1 p + \cdots)^{-1}$, then we have

$$\begin{aligned}\int_{\mathbb{Q}_p} f(|x|_p)\psi(\xi x)dx &= \int_{\mathbb{Q}_p} f(|\eta x|_p)\psi(|\xi|_p^{-1}x)d(\eta x)\\ &= \int_{\mathbb{Q}_p} f(|x|_p)\psi(|\xi|_p^{-1}x)dx\\ &= \sum_{\gamma\in\mathbb{Z}} f(p^\gamma)\int_{S_\gamma}\psi(|\xi|_p^{-1}x)dx.\end{aligned}$$

□

REMARKS.

(1) In (2.1), $|\xi|_p^{-1}$ denotes not only real number but also $p-$adic number such that $|\xi|_p^{-1}(1 + 0p + 0p^2 + \cdots)$.

(2) (2.1) states that if f is a function of norm $|x|_p$, then the integral in the right hand side is a function of norm $|\xi|_p$. Hence if we replace ψ by χ_p in (2.1), then Fourier transform $F(\xi)$ of $f(|x|_p)$ can be obtained as follows :

$$\begin{aligned}F(\xi) &= \sum_{\gamma\in\mathbb{Z}} f(p^\gamma)\int_{S_\gamma}\chi_p(\xi x)dx\\ &= \sum_{\gamma\in\mathbb{Z}} f(p^\gamma)\lambda(\xi,\gamma),\end{aligned} \tag{2.2}$$

where

$$\lambda(\xi,\gamma) \overset{\text{def}}{=} \begin{cases} p^{\gamma}(1-\frac{1}{p}), & \text{if } |\xi|_p \le p^{-\gamma}, \\ -p^{\gamma-1}, & \text{if } |\xi|_p = p^{-\gamma+1}, \\ 0, & \text{if } |\xi|_p \ge p^{-\gamma+2}, \quad \gamma \in \mathbb{Z} \end{cases} \tag{2.3}$$

and hence $F(\xi)$ is a function of norm $|\xi|_p$ ([1]).

Definition. For a given $g \in L^2(\mathbb{Q}_p)$, the mapping $f \hookrightarrow F_g f$, defined by

$$(F_g f)(\xi,q) \overset{\text{def}}{=} \frac{1}{\|g\|_2} \int_{\mathbb{Q}_p} f(x)\bar{g}(x-q)\chi_p(\xi x)dx, \tag{2.4}$$

is called the windowed Fourier transform from $L^2(\mathbb{Q}_p)$ into $L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$.

The inverse windowed Fourier transform in $\mathbb{Q}_p$ may be obtained from the same procedure as in $\mathbb{R}$, and hence we have

$$f(x) = \frac{1}{\|g\|_2} \int_{\mathbb{Q}_p} (F_g f)(\xi,q) g(x-q)\chi_p(\xi x) d\xi dq$$

under the condition that the integral in the right hand side exists.

In the sequel we shall need the following integrals :

Proposition 2. *We have*

$$\begin{aligned} \lambda(\xi,\gamma;k_0) &\overset{\text{def}}{=} \int_{S_\gamma, x_0=k_0} \chi_p(|\xi|_p^{-1}x)dx \\ &= \begin{cases} \chi_p(|\xi|_p^{-1}p^{\gamma}k)p^{\gamma-1}, & \textit{if } |\xi|_p \le p^{-\gamma+1} \\ 0, & \textit{if } |\xi|_p \ge p^{-\gamma+2} \end{cases} \end{aligned} \tag{2.5a}$$

$$\begin{aligned} &\lambda(\xi,\gamma;k_0,\cdots,k_l) \\ &\overset{\text{def}}{=} \int_{S_\gamma, x_0=k_0,\cdots,x_l=k_l} \chi_p(|\xi|_p^{-1}x)dx \\ &= \begin{cases} \chi_p(|\xi|_p^{-1}p^{-\gamma}(k_0+\cdots+k_l p^l))p^{\gamma-l-1}, & \textit{if } |\xi|_p \le p^{\gamma-l-1} \\ 0, & \textit{if } |\xi|_p \ge p^{\gamma-l} \end{cases} \end{aligned} \tag{2.5b}$$

Proof. (2.5*a*) is a particular case of (2.5*b*) when $l = 0$. (2.5*b*) can be easily proved by the direct computation. □

THEOREM. *Let $f \in L^2(\mathbb{Q}_p)$ be a (step) function defined by , for $x = |x|_p^{-1}(x_0 + x_1 p + \cdots)$,*

$$f(x) = f(k|x|_p^{-1}), \text{ if } x_0 = k, \ 1 \le k \le p-1 \tag{2.6}$$

and let $g \in L^2(\mathbb{Q}_p)$ be a function of norm $|x|_p$. Then we have

(2.7)

$$\begin{aligned}
&(F_g f)(\xi, q) \\
&= \frac{1}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma > \gamma_q} f(kp^{-\gamma}) \bar{g}(p^\gamma) \lambda(\xi, \gamma; k) \\
&+ \frac{\bar{g}(|q|_p)}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma < \gamma_q} f(k|x|_p^{-1}) \lambda(\xi, \gamma; k) \\
&+ \frac{f(q_0 p^{-\gamma_q})}{\|g\|_2} \sum_{k=0}^{\infty} \bar{g}(p^{\gamma_q - k}) [\lambda(\xi, \gamma_q; q_0, \cdots, q_{k-1}) - \lambda(\xi, \gamma_q; q_0, \cdots, q_k)]
\end{aligned}$$

where γ and γ_q denote integers such that $|x|_p = p^\gamma$ and $|q|_p = p^{\gamma_q}$ respectively.

Proof. We may have

$$\begin{aligned}
(F_g f)(\xi, q) &= \frac{1}{\|g\|_2} \int_{\mathbb{Q}_p} f(x) \bar{g}(x - q) \chi_p(\xi x) dx \\
&\stackrel{\text{def}}{=} I_1 + I_2 + I_3,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
I_1 &\stackrel{\text{def}}{=} \frac{1}{\|g\|_2} \int_{|x|_p > |q|_p} f(x) \bar{g}(|x|_p) \chi_p(\xi x) dx, \\
I_2 &\stackrel{\text{def}}{=} \frac{\bar{g}(|q|_p)}{\|g\|_2} \int_{|x|_p < |q|_p} f(x) \chi_p(\xi x) dx, \\
I_3 &\stackrel{\text{def}}{=} \frac{1}{\|g\|_2} \int_{|x|_p = |q|_p} f(x) \bar{g}(|x - q|_p) \chi_p(\xi x) dx.
\end{aligned} \tag{2.9}$$

For I_1, given $\xi = |\xi|_p^{-1}(\xi_0 + \xi_1 p + \cdots)$ in the canonical form, if we put $\xi' \stackrel{\text{def}}{=} (\xi_0 + \xi_1 p + \cdots)^{-1} \stackrel{\text{def}}{=} \xi'_0 + \xi'_1 p + \cdots$ and change variables by $(\xi_0 + \xi_1 p + \cdots)x = x'$, then we have

$$
\begin{aligned}
I_1 &= \frac{1}{\|g\|_2} \int_{|x|_p > |q|_p} f(\xi' x') \bar{g}(|\xi' x'|_p) \chi_p(|\xi|_p^{-1} x') d(\xi' x') \\
&= \frac{1}{\|g\|_2} \int_{|x|_p > |q|_p} f(\xi' x) \bar{g}(|x|_p) \chi_p(|\xi|_p^{-1} x) dx.
\end{aligned} \tag{2.10}
$$

If we write $\xi' x = |x|_p^{-1}(x'_0 + x'_1 p + \cdots)$ in the canonical form, then $x'_0 \equiv \xi'_0 x_0 (\text{mod } p)$. Since $(\xi'_0, p) = 1$, each x'_0 determines uniquely x_0 and vice versa. Hence we have

$$
\begin{aligned}
I_1 &= \frac{1}{\|g\|_2} \sum_{k=1}^{p-1} \int_{|x|_p > |q|_p, x_0 = k} f(k|x|_p^{-1}) \bar{g}(|x|_p) \chi_p(|\xi|_p^{-1} x) dx \\
&= \frac{1}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma > \gamma_q} f(kp^{-\gamma}) \bar{g}(p^{\gamma}) \int_{S_\gamma, x_0 = k} \chi_p(|\xi|_p^{-1} x) dx \\
&= \frac{1}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma > \gamma_q} f(kp^{-\gamma}) \bar{g}(p^{\gamma}) \lambda(\xi, \gamma; k).
\end{aligned} \tag{2.11}
$$

For I_2, by the same way as for I_1, we have

$$
\begin{aligned}
I_2 &= \frac{1}{\|g\|_2} \int_{|x|_p < |q|_p} f(\xi' x') \bar{g}(|q|_p) \chi_p(|\xi|_p^{-1} x') d(\xi' x') \\
&= \frac{\bar{g}(|q|_p)}{\|g\|_2} \sum_{k=1}^{p-1} \int_{|x|_p < |q|_p, x_0 = k} f(k|x|_p^{-1}) \chi_p(|\xi|_p^{-1} x) dx \\
&= \frac{\bar{g}(|q|_p)}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma < \gamma_q} f(kp^{-\gamma}) \int_{S_\gamma, x_0 = k} \chi_p(|\xi|_p^{-1} x) dx \\
&= \frac{\bar{g}(|q|_p)}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma < \gamma_q} f(kp^{-\gamma}) \lambda(\xi, \gamma; k).
\end{aligned} \tag{2.12}
$$

For I_3, we may have

$$I_3 = \frac{1}{\|g\|_2} \int_{S_{\gamma_q}} f(x)\bar{g}(|x-q|_p)\chi_p(\xi x)dx, \tag{2.13}$$

where

(2.14)

$$\begin{aligned}
&\int_{S_{\gamma_q}} f(x)\bar{g}(|x-q|_p)\chi_p(\xi x)dx \\
&= \sum_{k=0}^{\infty} \int_{S_{\gamma_q}, x_0=q_0,\cdots,x_{k-1}=q_{k-1},x_k\neq q_k} f(x)\bar{g}(|x-q|_p)\chi_p(\xi x)dx \\
&= f(q_0 p^{-\gamma_q}) \sum_{k=0}^{\infty} \bar{g}(||q|_p^{-1}p^k|_p) \int_{S_{\gamma_q}, x_0=q_0,\cdots,x_{k-1}=q_{k-1},x_k\neq q_k} \chi_p(\xi x)dx \\
&= f(q_0 p^{-\gamma_q}) \sum_{k=0}^{\infty} \bar{g}(p^{\gamma_q-k}) \int_{S_{\gamma_q}, x_0=q_0,\cdots,x_{k-1}=q_{k-1},x_k\neq q_k} \chi_p(\xi x)dx \\
&= f(q_0 p^{-\gamma_q}) \sum_{k=0}^{\infty} \bar{g}(p^{\gamma_q-k}) \left(\lambda(\xi,\gamma_q;q_0,\cdots,q_{k-1}) - \lambda(\xi,\gamma_q;q_0,\cdots,q_k)\right)
\end{aligned}$$

Substituting (2.14) into (2.13) and combining (2.11), (2.12) and (2.13), we complete our proof. □

Corollary 1. *Let $f, g \in L^2\mathbb{Q}_p$ be a function of $|x|_p$, then*

$$\begin{aligned}
(F_g f)(\xi,q) = &\frac{1}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma>\gamma_q} f(p^{-\gamma})\bar{g}(p^{\gamma})\lambda(\xi,\gamma;k) \\
&+ \frac{\bar{g}(|q|_p)}{\|g\|_2} \sum_{k=1}^{p-1} \sum_{\gamma<\gamma_q} f(p^{-\gamma})\lambda(\xi,\gamma;k) \\
&+ \frac{f(p^{-\gamma_q})}{\|g\|_2} \sum_{k=0}^{\infty} \bar{g}(p^{\gamma_q-k})[\lambda(\xi,\gamma_q;q_0,\cdots,q_{k-1}) \\
&\qquad - \lambda(\xi,\gamma_q;q_0,\cdots,q_k)]
\end{aligned} \tag{2.15}$$

We may write, from (2.4),

$$(F_g f)(\xi,q) = \frac{\chi_p(\xi q)}{\|g\|_2} \int_{\mathbb{Q}_p} f(x-(-q))\bar{g}(x)\chi_p(\xi x)dx.$$

Hence, by replacing $-q$ by q and interchanging the roles of f and $\bar{g}$ in the proof of Theorem, we obtain the following theorem :

Corollary 2. *Let g be a step function defined by* (2.6) *and f be a function of norm $|x|_p$, then we have*

$$\begin{aligned}
&\bar{\chi}_p(\xi q)(F_g f)(\xi, q)\\
&= \frac{1}{\|g\|_2}\sum_{k=1}^{p-1}\sum_{\gamma>\gamma_q}\bar{g}(kp^{-\gamma})f(p^{\gamma})\lambda(\xi,\gamma;k)\\
&+ \frac{f(|q|_p)}{\|g\|_2}\sum_{k=1}^{p-1}\sum_{\gamma<\gamma_q}\bar{g}(k|x|_p^{-1})\lambda(\xi,\gamma;k) \qquad (2.16)\\
&+ \frac{\bar{g}(q_0p^{-\gamma_q})}{\|g\|_2}\sum_{k=0}^{\infty}\bar{f}(p^{\gamma_q-k})[\lambda(\xi,\gamma_q;q_0,\cdots,q_{k-1})\\
&\qquad\qquad - \lambda(\xi,\gamma_q;q_0,\cdots,q_k)].
\end{aligned}$$

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