Kangweon-Kyungki Math. Jour. **13** (2005), No. 1, pp. 1–11

THE EXISTENCE AND UNIQUENESS OF E(*k)-CONNECTION IN n-$^*g$-UFT

Jong Woo Lee

Abstract. The purpose of the present paper is to introduce a new concept of the E(*k)-connection $\Gamma^{\nu}_{\lambda\mu}$, which is both Einstein and (*k)-connection, and to obtain a necessary and sufficient condition for the existence of the unique E(*k)-connection in n-$^*g$-UFT. Next, under this condition, we shall obtain a surveyable tensorial representation of the unique E(*k)-connection in n-$^*g$-UFT.

1. Introduction

Einstein[1] proposed a new unified field theory that would include both gravitation and electromagnetism. Characterizing Einstein's unified field theory as a set of geometrical postulates in the space-time X_4, Hlavatý[10] gave its mathematical foundation for the first time, and generalized X_4 to the n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, the so-called *Einstein's n-dimensional unified field theory(n-g-UFT)*. Since then many consequences of this theory has been obtained. In particular, the representations of the Einstein connection satisfying Einstein's equations in n-g-UFT, imposing some conditions to X_n, were obtained by Chung[6] and Lee[2, 3]. Corresponding to n-g-UFT, Chung[7, 8] introduced a new unified field theory, called *Einstein's n-dimensional *g-unified field theory(n-$^*g$-UFT)*. This theory is more useful than n-g-UFT in some physical aspects. Chung[7∼9] obtained many consequences of this theory. In n-$^*g$-UFT, however, it has been unable yet to represent a general n-dimensional Einstein's connection in a surveyable tensorial

Received September 17, 2004.
2000 Mathematics Subject Classification: 53A45, 53B50, 53C25.
Key words and phrases: n-g-UFT, n-$^*g$-UFT, Einstein's equation, Einstein connection, (*k)-connection, E(*k)-connection.
This study was supposed by the Grant of Maeji Institute of Academic Research, Yonsei University, 2002-2004.

form. In n-$^*g$-UFT, a connection $\Gamma^{\nu}_{\lambda\mu}$ which is both Einstein and (*k)-connection is called an E(*k)-connection. The purpose of the present paper is to obtain a necessary and sufficient condition for the existence of the unique E(*k)-connection in n-$^*g$-UFT. Next, under this condition, we shall obtain a precise tensorial representation of the unique E(*k)-connection. The obtained results and discussions in the present paper will be useful for the n-dimensional considerations of the unified field theory.

2. Preliminaries

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^{\nu}\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \cdots, n\}$ and follow the summation convention. In the Einstein's usual n-dimensional unified field theory(n-g-UFT), the algebraic structure on X_n is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.1}$$

where we assume that

$$det(g_{\lambda\mu}) \neq 0, \quad det(h_{\lambda\mu}) \neq 0, \quad det(k_{\lambda\mu}) \neq 0. \tag{2.2}$$

Since $det(h_{\lambda\mu}) \neq 0$, we may define a unique tensor $h^{\lambda\nu}(= h^{\nu\lambda})$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{2.3}$$

We use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined in n-g-UFT in the usual manner. Then we may define new tensors by

$$g^{\lambda\mu} = g_{\alpha\beta} h^{\lambda\alpha} h^{\mu\beta}, \quad k^{\lambda\mu} = k_{\alpha\beta} h^{\lambda\alpha} h^{\mu\beta}, \quad k_{\lambda}{}^{\nu} = k_{\lambda\mu} h^{\mu\nu}, \tag{2.4}$$

so that in virtue of (2.1) and (2.3), we obtain

$$g^{\lambda\mu} = h^{\lambda\mu} + k^{\lambda\mu}. \tag{2.5}$$

It should be remarked that since $k_{\lambda\mu}$ is a skew-symmetric tensor and $det(k_{\lambda\mu}) \neq 0$, *n is even.* In n-g-UFT the differential geometric structure on X_n is imposed by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma^{\nu}_{\lambda\mu}$ defined by the Einstein's equations:

$$\partial_{\omega} g_{\lambda\mu} - g_{\alpha\mu}\Gamma^{\alpha}_{\lambda\omega} - g_{\lambda\alpha}\Gamma^{\alpha}_{\omega\mu} = 0 \quad (\partial_{\nu} = \frac{\partial}{\partial x^{\nu}}), \tag{2.6a}$$

or equivalently

$$D_{\omega} g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}, \tag{2.6b}$$

where D_{ω} denotes the symbolic vector of the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$, and $S_{\lambda\mu}{}^{\nu}$ is the torsion tensor of $\Gamma^{\nu}_{\lambda\mu}$.

But in our Einstein's n-dimensional $^*g$-unified field theory(n-$^*g$-UFT), the role of the basic tensor is no longer played by $g_{\lambda\mu}$. In n-$^*g$-UFT the algebraic structure on the same space X_n is imposed by the basic real non-symmetric tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta^{\nu}_{\mu}. \tag{2.7}$$

It may be also decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}, \tag{2.8}$$

where we assume that $det(^*h^{\lambda\nu}) \neq 0$. Therefore we may also define a unique tensor $^*h_{\lambda\mu}(= {}^*h_{\mu\lambda})$ by

$$^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{2.9}$$

We use both $^*h^{\lambda\nu}$ and $^*h_{\lambda\mu}$ *as tensors for raising and/or lowering indices for all tensors defined in* n-$^*g$-*UFT in the usual manner.* Then we may also define new tensors by

$$\begin{aligned} &^*g_{\lambda\mu} = {}^*g^{\alpha\beta}\,{}^*h_{\lambda\alpha}\,{}^*h_{\mu\beta}, \\ &^*k_{\lambda\mu} = {}^*k^{\alpha\beta}\,{}^*h_{\lambda\alpha}\,{}^*h_{\mu\beta}, \quad {}^*k_{\lambda}{}^{\nu} = {}^*k^{\alpha\nu}\,{}^*h_{\alpha\lambda}, \end{aligned} \tag{2.10}$$

so that in virtue of (2.8) and (2.9) we obtain

$$ {}^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu}. \tag{2.11}$$

On the other hand, in n-${}^*g$-UFT the differential geometrical structure on X_n is imposed by the tensor ${}^*g^{\lambda\nu}$ by means of a connection $\Gamma^\nu_{\lambda\mu}$ defined by a system of ${}^*g$-Einstein's equations:

$$ \partial_\omega {}^*g^{\lambda\nu} + {}^*g^{\alpha\nu}\Gamma^\lambda_{\alpha\omega} + {}^*g^{\lambda\alpha}\Gamma^\nu_{\omega\alpha} = 0, \tag{2.12a}$$

or equivalently

$$ D_\omega {}^*g^{\lambda\nu} = -2S_{\omega\alpha}{}^{\nu}\, {}^*g^{\lambda\alpha}. \tag{2.12b}$$

Hlavatý[10] proved that the system of ${}^*g$-Einstein's equations (2.12) is equivalent to the system of original Einstein's equations (2.6).

The following quantities are frequently used in our further considerations: For every integer $p \geq 1$,

$$ {}^{(0)*}k_\lambda{}^\nu = \delta^\nu_\lambda, \quad {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha\, {}^{(p-1)*}k_\alpha{}^\nu = {}^{(p-1)*}k_\lambda{}^\alpha\, {}^*k_\alpha{}^\nu. \tag{2.13}$$

It should be remarked that the tensor ${}^{(p)*}k_{\lambda\nu}$ is symmetric if p is even, and skew-symmetric if p is odd.

3. Existence of E(*k)-connection

Agreement 3.1. All our further considerations in the present paper are dealt in n-${}^*g$-UFT, where n is even. □

Definition 3.2. A connection $\Gamma^\nu_{\lambda\mu}$ is said to be *Einstein* if it satisfies the system of ${}^*g$-Einstein's equations (2.12). A connection $\Gamma^\nu_{\lambda\mu}$ is said to be *(*k)-connection* if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form

$$ S_{\lambda\mu}{}^\nu = {}^*k_{\lambda\mu}Y^\nu, \tag{3.1}$$

for some nonzero vector Y^ν. A connection which is both Einstein and (*k)-connection is called an *E(*k)-connection.*

THEOREM 3.3. *when for some nonzero vector Y^ν the condition (3.1) holds, the system of equations (2.12) is equivalent to the following system of equations:*

$$D_\omega {}^*h^{\lambda\nu} = -2\, {}^*k_\omega{}^{(\lambda} Y^{\nu)} + 2\, {}^{(2)*}k_\omega{}^{(\lambda} Y^{\nu)}, \tag{3.2a}$$

$$D_\omega {}^*k^{\lambda\nu} = -2\, {}^*k_\omega{}^{[\lambda} Y^{\nu]} + 2\, {}^{(2)*}k_\omega{}^{[\lambda} Y^{\nu]}. \tag{3.2b}$$

Proof. Substituting (2.8) and (3.1) into (2.12b), we obtain

$$D_\omega {}^*g^{\lambda\nu} = -2\, {}^*k_\omega{}^{\lambda} Y^{\nu} + 2\, {}^{(2)*}k_\omega{}^{\lambda} Y^{\nu}. \tag{3.3}$$

The equations (3.2a) and (3.2b) follow from (3.3) and from

$$D_\omega {}^*h^{\lambda\nu} = D_\omega {}^*g^{(\lambda\nu)}, \qquad D_\omega {}^*k^{\lambda\nu} = D_\omega {}^*g^{[\lambda\nu]}.$$

Conversely, taking the sum of (3.2a) and (3.2b), we obtain (3.3). □

THEOREM 3.4. *The equation (3.2a) is equivalent to the following equation:*

$$D_\omega {}^*h_{\lambda\mu} = 2\, {}^*k_{\omega(\lambda} Y_{\mu)} - 2\, {}^{(2)*}k_{\omega(\lambda} Y_{\mu)}. \tag{3.4}$$

Proof. Differentiating (2.9) covariantly with respect to $\Gamma^\nu_{\lambda\mu}$, we obtain

$$D_\omega {}^*h_{\lambda\mu} = -{}^*h_{\alpha\mu} {}^*h_{\beta\lambda} (D_\omega {}^*h^{\alpha\beta}), \tag{3.5a}$$

$$D_\omega {}^*h^{\lambda\nu} = -{}^*h^{\alpha\nu}\, {}^*h^{\beta\lambda} (D_\omega {}^*h_{\alpha\beta}). \tag{3.5b}$$

Substituting (3.2a) into (3.5a), and using (2.10), we obtain (3.4). Conversely, substituting (3.4) into (3.5b), and using (2.10), we obtain (3.2a). □

THEOREM 3.5. *when for some nonzero vector Y^ν the condition (3.1) holds, the system of equations (2.12) is equivalent to the followings:*

$$\Gamma^\nu_{\lambda\mu} = {}^*\{{}_\lambda{}^\nu{}_\mu\} + {}^{(2)*}k_{\lambda\mu} Y^\nu + {}^*k_{\lambda\mu} Y^\nu, \tag{3.6}$$

$$\nabla_\omega {}^*k^{\lambda\nu} = -2({}^*k_\omega{}^{[\lambda} - {}^{(3)*}k_\omega{}^{[\lambda}) Y^{\nu]}, \tag{3.7}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols ${}^*\{{}_\lambda{}^\nu{}_\mu\}$ defined by ${}^*h_{\lambda\mu}$.

Proof. From Theorem 3.3 and Theorem 3.4, when for some nonzero vector Y^ν the condition (3.1) holds, the system of equations (2.12) is equivalent to the system of equations (3.2b) and (3.4). In virtue of relation

$$D_\omega {}^*h_{\lambda\mu} = \partial_\omega {}^*h_{\lambda\mu} - {}^*h_{\alpha\mu}\Gamma^\alpha_{\lambda\omega} - {}^*h_{\lambda\alpha}\Gamma^\alpha_{\mu\omega}, \tag{3.8}$$

and (3.1), we obtain

$$\begin{aligned} &\frac{1}{2}{}^*h^{\nu\alpha}(D_\lambda {}^*h_{\alpha\mu} + D_\mu {}^*h_{\alpha\lambda} - D_\alpha {}^*h_{\lambda\mu}) \\ &= {}^*\{\lambda{}^\nu{}_\mu\} - 2S^\nu{}_{(\lambda\mu)} + S_{\lambda\mu}{}^\nu - \Gamma^\nu_{\lambda\mu} \\ &= {}^*\{\lambda{}^\nu{}_\mu\} + 2\,{}^*k_{(\lambda}{}^\nu Y_{\mu)} + {}^*k_{\lambda\mu}Y^\nu - \Gamma^\nu_{\lambda\mu}. \end{aligned} \tag{3.9a}$$

On the other hand, it follows from (3.4) that

$$\begin{aligned} &\frac{1}{2}{}^*h^{\nu\alpha}(D_\lambda {}^*h_{\alpha\mu} + D_\mu {}^*h_{\alpha\lambda} - D_\alpha {}^*h_{\lambda\mu}) \\ &= 2\,{}^*k_{(\lambda}{}^\nu Y_{\mu)} - {}^{(2)*}k_{\lambda\mu}Y^\nu. \end{aligned} \tag{3.9b}$$

Comparing (3.9a) with (3.9b), we obtain (3.6). On the other hand, substituting (3.6) into

$$D_\omega {}^*k^{\lambda\nu} = \partial_\omega {}^*k^{\lambda\nu} + {}^*k^{\alpha\nu}\Gamma^\lambda_{\alpha\omega} + {}^*k^{\lambda\alpha}\Gamma^\nu_{\alpha\omega},$$

we obtain

$$D_\omega {}^*k^{\lambda\nu} = \nabla_\omega {}^*k^{\lambda\nu} - 2\,{}^{(3)*}k_\omega{}^{[\lambda}Y^{\nu]} + 2\,{}^{(2)*}k_\omega{}^{[\lambda}Y^{\nu]}. \tag{3.10}$$

Comparing (3.2b) with (3.10), we obtain (3.7). Conversely, suppose that (3.6) and (3.7) hold. Substituting (3.6) into (3.8), we obtain (3.4). Similarly, substituting (3.7) into (3.10), we obtain (3.2b). □

4. Uniqueness of E(*k)-connection

Remark 4.1. In virtue of Theorem 3.5, it is obvious that if the system of equations (2.12) admits an E(*k)-connection $\Gamma^\nu_{\lambda\mu}$, it must be of the form (3.6). This reduces the investigation of an E(*k)-connection $\Gamma^\nu_{\lambda\mu}$ to the study of the vector Y^ν defining (3.6). In order to know the E(*k)-connection $\Gamma^\nu_{\lambda\mu}$ it is necessary and sufficient to know the vector Y^ν satisfying the equation (3.7), which is the main goal of this section. Our investigation is based on the skew-symmetric tensor

$${}^*P^{\lambda\nu} = {}^*k^{\lambda\nu} - {}^{(3)*}k^{\lambda\nu}. \tag{4.1}$$

Lemma 4.2. *For every integer $p \geq 1$, the tensor ${}^{(p)*}k^{\lambda\nu}$ satisfies the following relations:*

(4.2a)
$$ {}^{(p)*}k^{\lambda\nu} g_{\lambda\mu} = \sum_{f=1}^{p} (-1)^{f-1}\ {}^{(p-f)*}k_{\mu}{}^{\nu} + (-1)^{p}\ {}^{*}h^{\lambda\nu} g_{\lambda\mu}, $$

(4.2b)
$$ {}^{(p)*}k^{\lambda\nu} g_{\mu\lambda} = -\sum_{f=1}^{p} {}^{(p-f)*}k_{\mu}{}^{\nu} + {}^{*}h^{\lambda\nu} g_{\mu\lambda}. $$

Proof. This assertion (4.2a) will be proved by induction on p. Substituting (2.8) into (2.7), we obtain

$$ {}^{*}k^{\lambda\nu} g_{\lambda\mu} = \delta_{\mu}^{\nu} - {}^{*}h^{\lambda\nu} g_{\lambda\mu}. \tag{4.3} $$

Hence in virtue of (2.13), the assertion (4.2a) holds for the case $p = 1$. Now, assume that (4.2a) is true for the case $p = m$, i.e.,

$$ {}^{(m)*}k^{\lambda\nu} g_{\lambda\mu} = \sum_{f=1}^{m} (-1)^{f-1}\ {}^{(m-f)*}k_{\mu}{}^{\nu} + (-1)^{m}\ {}^{*}h^{\lambda\nu} g_{\lambda\mu}. \tag{4.4} $$

Multiplying ${}^{*}k_{\nu}{}^{\omega}$ to both sides of (4.4), and using (2.13) and (4.3), we obtain

$$ \begin{aligned} {}^{(m+1)*}k^{\lambda\omega} g_{\lambda\mu} &= \sum_{f=1}^{m} (-1)^{f-1}\ {}^{(m-f+1)*}k_{\mu}{}^{\omega} + (-1)^{m}\ {}^{*}k^{\lambda\omega} g_{\lambda\mu} \\ &= \sum_{f=1}^{m} (-1)^{f-1}\ {}^{(m-f+1)*}k_{\mu}{}^{\omega} + (-1)^{m}\ \delta_{\mu}^{\omega} + (-1)^{m+1}\ {}^{*}h^{\lambda\omega} g_{\lambda\mu} \\ &= \sum_{f=1}^{m+1} (-1)^{f-1}\ {}^{(m+1-f)*}k_{\mu}{}^{\omega} + (-1)^{m+1}\ {}^{*}h^{\lambda\omega} g_{\lambda\mu}, \end{aligned} $$

which shows that (4.2a) holds for the case $p = m+1$. By the principle of induction, the assertion (4.2a) is true for every integer $p \geq 1$. Similarly, we obtain (4.2b). □

Lemma 4.3. *The following relation holds*

$$h^{\lambda\nu} = {}^*h^{\lambda\nu} - {}^{(2)*}k^{\lambda\nu}. \tag{4.5}$$

Proof. When $p = 2$, (4.2a) and (4.2b) satisfy the following relations:

$$({}^*h^{\lambda\nu} - {}^{(2)*}k^{\lambda\nu})g_{\lambda\mu} = -{}^*k_\mu{}^\nu + \delta^\nu_\mu, \tag{4.6a}$$

$$({}^*h^{\lambda\nu} - {}^{(2)*}k^{\lambda\nu})g_{\mu\lambda} = {}^*k_\mu{}^\nu + \delta^\nu_\mu. \tag{4.6b}$$

Taking the sum of (4.6a) and (4.6b), and using (2.1), we obtain

$$({}^*h^{\lambda\nu} - {}^{(2)*}k^{\lambda\nu})h_{\lambda\mu} = \delta^\nu_\mu,$$

which implies (4.5) in virtue of (2.3). □

Theorem 4.4. *The determinant of the tensor* ${}^*P^{\lambda\nu}$*, given by (4.1), never vanishes, i.e.,*

$$det({}^*P^{\lambda\nu}) \neq 0. \tag{4.7}$$

Proof. Subtracting (4.6b) from (4.6a), and using (2.1), we obtain

$$({}^*h^{\lambda\nu} - {}^{(2)*}k^{\lambda\nu})k_{\lambda\mu} = -{}^*k_\mu{}^\nu. \tag{4.8}$$

Using (2.4), (4.5) and (4.8), we obtain

$$k_\mu{}^\nu = -h^{\lambda\nu}k_{\lambda\mu} = -({}^*h^{\lambda\nu} - {}^{(2)*}k^{\lambda\nu})k_{\lambda\mu} = {}^*k_\mu{}^\nu. \tag{4.9}$$

Next, using (2.4), (2.13), (4.5) and (4.9), we obtain

$$k^{\lambda\nu} = h^{\lambda\alpha}k_\alpha{}^\nu = ({}^*h^{\lambda\alpha} - {}^{(2)*}k^{\lambda\alpha}){}^*k_\alpha{}^\nu = {}^*k^{\lambda\nu} - {}^{(3)*}k^{\lambda\nu} = {}^*P^{\lambda\nu}.$$

From which it follows that in virtue of (2.4),

$$\begin{aligned} det({}^*P^{\lambda\nu}) &= det(k^{\lambda\nu}) = det(h^{\lambda\alpha}\, k_{\alpha\beta}\, h^{\beta\mu}) \\ &= det(h^{\lambda\alpha})det(k_{\alpha\beta})det(h^{\beta\mu}). \end{aligned}$$

Since $det(h^{\lambda\nu}) \neq 0$ and $det(k_{\alpha\beta}) \neq 0$, we obtain (4.7). □

REMARK 4.5. Since $det({}^*P^{\lambda\nu}) \neq 0$, there is a unique skew-symmetric tensor ${}^*Q_{\lambda\mu}$ satisfying

$$ {}^*P^{\lambda\nu}\,{}^*Q_{\lambda\mu} = \delta^{\nu}_{\mu}. \tag{4.10}$$

THEOREM 4.6. *A necessary and sufficient condition for the system (2.12) to admit exactly one E(*k)-connection* $\Gamma^{\nu}_{\lambda\mu}$ *of the form (3.6) is that the basic tensor* ${}^*g^{\lambda\nu}$ *satisfies the following condition:*

$$\nabla_{\omega}{}^*k^{\lambda\nu} = -2\,{}^*P_{\omega}{}^{[\lambda}\,{}^*Q_{\alpha}{}^{\nu]}\,\nabla_{\beta}{}^*k^{\alpha\beta}, \tag{4.11}$$

where ${}^*P^{\lambda\nu}$ *is defined by (4.1), and* ${}^*Q_{\lambda\mu}$ *by (4.10). If this condition is satisfied, then the vector* Y^{ν} *which defines the E(*k)-connection is given by*

$$Y^{\alpha} = {}^*Q_{\lambda}{}^{\alpha}\,\nabla_{\beta}{}^*k^{\lambda\beta}. \tag{4.12}$$

Proof. If the system (2.12) admits a solution of the form (3.6), then the condition (3.7) holds in virtue of Theorem 3.5. Using (4.1), the condition (3.7) is equivalent to

$$\nabla_{\omega}{}^*k^{\lambda\nu} = -2\,{}^*P_{\omega}{}^{[\lambda}\,Y^{\nu]}. \tag{4.13}$$

Contracting for ω and ν in (4.13), and using the skew-symmetry of ${}^*P^{\lambda\nu}$, we obtain

$$\nabla_{\beta}{}^*k^{\lambda\beta} = -{}^*P_{\beta}{}^{\lambda}\,Y^{\beta}. \tag{4.14}$$

Multiplying ${}^*Q_{\lambda}{}^{\alpha}$ on both sides of (4.14), we obtain (4.12) in virtue of (4.10). Substituting (4.12) into (4.13), we obtain (4.11). Conversely, suppose that the condition (4.11) holds. With the vector Y^{ν} given by (4.12), define a (*k)-connection by (3.6), and substitute this connection into (2.12). This connection satisfies (2.12) in virtue of our assumption (4.11). Hence it is Einstein. Therefore there exists an E(*k)-connection $\Gamma^{\nu}_{\lambda\mu}$. Assume now that there exist two E(*k)-connections $\Gamma^{\nu}_{\lambda\mu}$ and $\overline{\Gamma}^{\nu}_{\lambda\mu}$:

$$\begin{aligned} \Gamma^{\nu}_{\lambda\mu} &= {}^*\{_{\lambda}{}^{\nu}{}_{\mu}\} + {}^{(2)*}k_{\lambda\mu}Y^{\nu} + {}^*k_{\lambda\mu}Y^{\nu}, \\ \overline{\Gamma}^{\nu}_{\lambda\mu} &= {}^*\{_{\lambda}{}^{\nu}{}_{\mu}\} + {}^{(2)*}k_{\lambda\mu}\overline{Y}^{\nu} + {}^*k_{\lambda\mu}\overline{Y}^{\nu} \qquad (\overline{Y}^{\nu} \neq Y^{\nu}). \end{aligned} \tag{4.15}$$

Then in virtue of the proof of Theorem 3.5, Y^ν and $\overline{Y}^\nu$ must satisfy

$$-2\,{}^*P_\omega{}^{[\lambda}\,Y^{\nu]} = \nabla_\omega{}^*k^{\lambda\nu} = -2\,{}^*P_\omega{}^{[\lambda}\,\overline{Y}^{\nu]}. \tag{4.16}$$

Applying the same method used to derive (4.12), we have from (4.16)

$$Y^\alpha = {}^*Q_\lambda{}^\alpha\,\nabla_\beta{}^*k^{\lambda\beta} = \overline{Y}^\alpha,$$

which contradicts to the assumption (4.15). This proves the uniqueness of the E(*k)-connection under condition (4.11). □

THEOREM 4.7. *If the condition (4.11) is always satisfied by the basic tensor* ${}^*g^{\lambda\nu}$*, then the unique E(*k)-connection* $\Gamma^\nu_{\lambda\mu}$ *is represented as*

$$\begin{aligned}\Gamma^\nu_{\lambda\mu} &= {}^*\{{}_\lambda{}^\nu{}_\mu\} + ({}^{(2)*}k_{\lambda\mu} + {}^*k_{\lambda\mu}){}^*Q_\alpha{}^\nu\,\nabla_\beta{}^*k^{\alpha\beta}\\ &= {}^*\{{}_\lambda{}^\nu{}_\mu\} + {}^*k_\lambda{}^\omega\,{}^*g_{\omega\mu}\,{}^*Q_\alpha{}^\nu\,\nabla_\beta{}^*k^{\alpha\beta}.\end{aligned} \tag{4.17}$$

Proof. Substituting (4.12) into (3.6), we obtain (4.17). □

REMARK 4.8. The unique E(*k)-connection (4.17) which is obtained in the present paper will be useful for the n-dimensional considerations of the unified field theory. In particular, applying the similar method[4, 5] used in n-g-UFT, we shall be able to obtain a particular solution and an algebraic solution of ${}^*g$-Einstein's field equation in n-${}^*g$-UFT.

References

1. A. Einstein, *The meaning of relativity*, Princeton University Press, Princeton, New Jersey, 1950.
2. J. W. Lee, *Field equations on SE(k)-manifold X_n*, Inter. Jour. of Theor. Phys. **30** (1991), 1343–1353.
3. ______, *Geometric studies of n-dimensional E(k)-manifold*, Tensor, N. S. **51** (1992), 78–82.
4. ______, *An algebraic solution of Einstein's field equations*, Tensor, N. S. **61** (1999), 240–244.
5. ______, *A particular solution of Einstein's field equations*, Tensor, N. S. **61** (1999), 245–250.

6. K. T. Chung and C. H. Cho, *On the n-dimensional SE-connection and its conformal change*, Nuovo Cimento **100** (1987), 537–550.
7. K. T. Chung, *Einstein's connection in terms of* $^*g^{\lambda\nu}$, Nuovo Cimento **27** (1963), 1297–1324.
8. K. T. Chung and D. H. Cheoi, *Relations of two n-dimensional unified field theories*, Acta Math. Hung. **45** (1985), 141–149.
9. K. T. Chung and D. K. Jun, *On the geometry of the manifold* $*g$*-*SEX_n *and its conformal change*, Nuovo Cimento **106B** (1991c), 1271–1286.
10. V. Hlavatý, *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd., New York, 1957.

Department of Mathematics
Yonsei University
Wonju, 220-710, Korea
E-mail: jwlee@dragon.yonsei.ac.kr