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PRICING AND HEDGING OPTIONS IN AN EMPIRICAL ASSET MODEL

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Abstract. Pricing and hedging strategy for European options of jump-type asset models, which are derived from a stochastic differential equations, are discussed.

1. Introduction

In this paper, we think a jump-type asset model derived from canonical stochastic differential equation(SDE). By this model, we discuss option pricing and hedging problems.

In [1], we met the range interval problem of the prices of European call option for the class of all possible measures which were equivalent to given probability P and the asset model $S_t; t \geq 0$ which was the solution of 1-dimensional SDE:

$$dS_t = S_{t-}[dZ_t + (e^{\Delta Z_t} - 1 - \Delta Z_t)], \tag{1.1}$$

and whose driving process was a jump-type semimartingale:

$$Z_t = bt + \int_0^t \int_{|z|\leq 1} z\tilde{N}_p(dz, ds) + \int_0^t \int_{|z|>1} zN_p(dz, ds). \qquad (1.2).$$

As pointed out in [1], this asset model is very realistic in view of some empirical sense if we look at the microstructure of stock price movements.

In this paper, we deal an option price and an upper bound of the range set of option prices. Further, we will think a hedging problem

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of option for this model. In section 2, we define our asset model by the solution of canonical SDE. Also we summarize the range sets of European call option prices for 1-dimensional model which is in [1]. In section 3, we think an option price for some equivalent measure Q, and get the upper bound of option price set. In section 4, we define value process, and study the hedging problems. In this section, we fix a measure Q, which is equivalent to given probability P and makes $e^{-rt}\xi_t$ a martingale. We are not use Girsanov transformation to define semimartingale for simplicity, and will omit the notation of Q.

2. Asset model

To study mathematical finance, in general, we define an asset model which is defined from the notion of one of two kinds of assumptions. If we think the prices and its derivatives for several periods, the return Z_t which is defined by $Z_t = \log S_t - \log S_{t-1}$ is more reasonable, but if we think for maturity(T) only, the return Z_t defined by $Z_t = (S_t - S_{t-1})/S_{t-1}$ is more reasonable(c.f. [2]). In [1], we know that our model is more realistic than the traditional model:

$$S_t = S_0 \exp(Z_t), \tag{2.1}$$

where Z_t is a corresponding return process. To study in mathematical logic, we will introduce our model from SDE theory.

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a filtration $\mathcal{F}_t; t \in [0, \infty)$ of a right continuous increasing family of sub-σ-fields of $\mathcal{F}$. Let $Z_t; t \geq 0$ be an R-valued cadlag semimartingale defined on $(\Omega, \mathcal{F}, P)$. For each Z_t, we define a Poisson point process N_p over $R \times (0, T]$ by

$$N_p(U, (s,t]) = \sum_{s<r\leq t} \mathcal{X}_U(\Delta Z_r), \quad \Delta Z_r = Z_r - Z_{r-},$$

where U is a Borel subset of R excluding $\{0\}$. Then there exists a unique predictable process $\hat{N}_p(U,(0,t])$ such that $\tilde{N}_p(U,(0,t]) = N_p(U,(0,t]) - \hat{N}_p(U,(0,t])$ is a localmartingale if $\inf_{z\in U} |z| > 0$.

Let $Z_t; t \geq 0$ be a jump-type semimartingale of the form:

$$Z_t = bt + \int_U z\tilde{N}_p(dz,t) + \int_{U^c} zN_p(dz,t), \tag{2.2}$$

where $U = \{z \in R; |z| \leq 1\}$, $N_p(dz, dt)$ is a Poisson random measure on $R \times [0, \infty)$ with intensity measure $\hat{N}_p(dz, dt) = \nu(dz)dt$. The measure ν is a Lévy measure on R^m satisfying $\nu(\{0\}) = 0$, $\nu(\{z; |z| > 1\}) < \infty$ and $\int_{|z|\leq 1} |z|^2 \nu(dz) < \infty$.

Let v be a Lipschitz continuous vector field from R to itself. Consider a canonical SDE driven by a vector field-valued semimartingale $Z_t v$:

$$dS_t = v(S_t) \diamond dZ_t. \tag{2.3}$$

Then we can define a flow $S_t; 0 \leq t \leq T$ as the solution of SDE (2.3):

$$S_t = S_0 + \int_0^t v(\xi_r) \diamond dZ_r. \tag{2.4}$$

If we represent this by using Itô integral, then we get

$$\begin{aligned} S_t = S_0 &+ \int_0^t bv(S_r)dr + \int_0^t v(S_{r-})dZ_d(r) \\ &+ \sum_{0<r\leq t} [\exp(\Delta Z_r v)(S_{0,r-}) - S_{0,r-} - \Delta Z_r v(S_{0,r-})] \\ = S_0 &+ \int_0^t bv(S_{0,r})dr \qquad (2.5) \\ &+ \int_0^t \int_U [\exp(zv)(S_{0,r}) - S_{0,r} - zv(S_{0,r})]\nu(dz)dr \\ &+ \int_0^t \int_U [\exp(zv)(S_{0,r-}) - S_{0,r-}]\tilde{N}_p(dz, dr) \\ &+ \int_0^t \int_{U^c} [\exp(zv)(S_{0,r-}) - S_{0,r-}]N_p(dz, dr) \end{aligned}$$

because of

$$\begin{aligned} \int_0^t v(S_{0,r-})dZ_d(r) = &\int_0^t \int_U zv(S_{0,r-})\tilde{N}_p(dz, dr) \\ &+ \int_0^t \int_{U^c} zv(S_{0,r-})N_p(dz, dr) \end{aligned}$$

and

$$\sum_{0<r\leq t}[\exp(\Delta Z_r v)(S_{0,r-}) - S_{0,r-} - \Delta Z_r v(S_{0,r-})]$$
$$= \int_0^t \int_R [\exp(\Delta Z_r v)(S_{0,r-}) - S_{0,r-} - \Delta Z_r v(S_{0,r-})] N_p(dz, dr),$$

where $R = U \cup U^c$ and v is a smooth vector field on R.

As our preliminary, let us think range set of option prices in canonical SDE model. If we put $v(x) = x$, this model is reduced to the model of [1]. Thus we get:

$$\begin{aligned} S_t = S_0 &+ \int_0^T bS_s ds + \int_0^T \int_U [e^z S_s - S_s - zS_s]\nu(dz)ds \quad (2.6) \\ &+ \int_0^T \int_U [e^z S_{s-} - S_{s-}]\tilde{N}_p(dz, ds) \\ &+ \int_0^T \int_{U^c} [e^z S_{s-} - S_{s-}] N_p(dz, ds). \end{aligned}$$

In the following, the time-horizon T will be considered finite. For a continuous function $g : (0, \infty) \to [0, \infty)$ and an exercise price $K \in R$, we think an $\mathcal{F}_T$-measurable random variable $h : \Omega \to [0, \infty)$ defined by $h = g(S_T)$. We fix the rate function of interest as $k(t) := rt$. Under a measure Q which is equivalent to given probability P and makes $e^{-rt}S_t$ martingales, we define the value $u_Q(S_0)$ of option $h = g(S_T)$ at time $t = 0$ with maturity T by

$$u_Q(S_0) := E_Q[e^{-rT} g(S_T)|\xi_0], \quad (2.7)$$

where E_Q is an expectation with respect to Q. Then, for all measures which are equivalent to given probability P and make $e^{-rt}S_t$ martingales, we get the range interval of option price $u(\xi_0)$:

$$[e^{-rT} g(e^{rT} S_0), \quad S_0).$$

Further the range set of prices is dense subset of above interval by the same terminology of [1].

3. Pricing options

Let us think the canonical SDE of the form:

$$dS_t = v(S_t) \diamond dZ_t. \tag{3.1}$$

Then we get the solution:

$$\begin{aligned} S_t = S_0 + \int_0^t bv(S_s)ds \qquad (3.2) \\ + \int_0^t \int_U [\exp(zv)(S_s) - S_s - zv(S_s)]\nu(dz)ds \\ + \int_0^t \int_U [\exp(zv)(S_{s-}) - S_{s-}]\tilde{N}_p(dz, ds) \\ + \int_0^t \int_{U^c} [\exp(zv)(S_{s-}) - S_{s-}]N_p(dz, ds). \end{aligned}$$

Then, for the payoff function $g : (0, \infty) \to [0, \infty)$ such that $g(x) = (x - K)_+$ where $K \in R$ is an exercise price, the price of European call option is defined as;

$$u_Q(0, S_0) = E_Q[e^{-rT} g(S_T)|S_0] \tag{3.3}$$

It is almost impossible to calculate the number of price $u_Q(0, S_0)$ exactly, because we can't find appropriate new probability measure. Thus we think under the probability Q which makes $e^{-rt}S_t$ martingale.

THEOREM 3.1. *For an European call option $h = g(S_T)$, the value $u_Q(0, S_0)$ at time $t = 0$ of h with maturity T is given by*

$$u_Q(0, S_0) = e^{-rT} A(x)^{-1} \int_0^\infty g(x)A(x)dx,$$

where $A(x) = \int_0^\infty \delta_0(y)p(x, y)dy$, $p(x, y)$ is a density of joint law of (S_T, S_0) such that $\log p(\cdot, \cdot)$ is C^1 with a differential which grows at most in a polynomial way at infinity, $\delta_0(y) = -H(y)\frac{\partial}{\partial y} \log p(x, y)$ and $H = I_{(x>0)} + c$, $c \in R$ is arbitrary constant.

Proof. . By the representation formula in the application of Malliavin calculus to Monte Carlo method(c.f. [3]), we get that:

$$E_Q[g(S_T)|S_0] = E_Q[g(S_T)\delta_0(S_0)]E_Q[\delta_0(S_0)]^{-1},$$

and

$$E_Q[g(S_T)\delta_0(S_0)] = \int\int g(x)\delta_0(y)p(x,y)dxdy.$$

Thus, we get (c.f. Remark 5 of [1]):

$$\begin{aligned} u_Q(0,S_0) &= e^{-rT}E_Q[g(S_T)|S_0] \\ &= \frac{e^{-rT}}{E_Q[\delta_0(S_0)]}\int_0^\infty\int_0^\infty g(x)\delta_0(y)p(x,y)dxdy. \end{aligned}$$

□

REMARK. If the payoff function g is C^1-function, we can get another representation formulas (c.f. [3]): $E^Q[g(S_T)\delta_0(S_0)]$ is given by

$$\int\int\{g(x)H(y)q(x,y) - g'(x)H(y)r(x,y)\}p(x,y)dxdy,$$

provide q and r satisfy

$$q + \frac{1}{p}\frac{\partial}{\partial x}(rp) = -\frac{\partial}{\partial y}(\log p).$$

Let us think a solution of canonical SDE (2.3):

$$\begin{aligned} S_t = S_0 &+ \int_0^t bv(S_s)ds + \int_0^t\int_U[\exp(zv)(S_s) - S_s - zv(S_s)]\nu(dz)ds \\ &+ \int_0^t\int_U[\exp(zv)(S_{s-}) - S_{-s}]\tilde{N}_p(dz,ds) \\ &+ \int_0^t\int_{U^c}[\exp(zv)(S_{-s}) - S_{-s}]N_p(dz,ds). \quad (3.4) \end{aligned}$$

To get more small range interval for option price $u_Q^i(0,S_0^i)$, we assume that $g(S_T) = S_T - K \geq 0$. Then we get a result by putting:

$$B(S_s) = bv(S_s) + \int_U[\exp(zv)(S_s) - S_s - zv(S_s)]\nu(dz).$$

THEOREM 3.2. *If we assume $B(S_t) \leq rS_{t-}$, then we get*

$$u_Q(0, S_0) \leq S_0 - e^{-rT}K + E_Q[G_T|\mathcal{F}_0], \tag{3.5}$$

where

$$\begin{aligned} G_T = &\int_0^T \int_U [\exp(zv)(S_{s-}) - S_{s-}]\tilde{N}_p(dz, ds) \\ &+ \int_0^T \int_{U^c} [\exp(zv)(S_{s-}) - S_{s-}]N_p(dz, ds). \end{aligned}$$

Proof. By the integration by parts, we get

$$\begin{aligned} e^{-rt}S_t &= S_0 + \int_0^t S_{s-}de^{-rt} + \int_0^t e^{-rt}dS_s \\ &= S_0 + \int_0^t S_{s-}(-re^{-rs})ds + \int_0^t e^{-es}dS_s. \end{aligned}$$

Therefore we get

$$\begin{aligned} e^{-rt}S_t = S_0 &- \int_0^t rS_{s-}e^{-rs}ds + \int_0^t e^{-rs}B(S_s)ds \\ &+ \int_0^t e^{-rs}\int_U [\exp(zv)(S_{s-}) - S_{s-}]\tilde{N}_p(dz, ds) \\ &+ \int_0^t e^{-rs}\int_{U^c} [\exp(zv)(S_{s-}) - S_{s-}]N_p(dz, ds). \end{aligned}$$

Thus, from the assumption, we get

$$S_t \leq e^{rt}(S_0 + G_t).$$

Therefore, we get

$$\begin{aligned} &S_T - K \leq e^{rT}(S_0 + G_T) - K \quad a.s., \\ &e^{-rT}(S_T - K) \leq S_0 + G_T - e^{-rT}K \quad a.s.. \end{aligned}$$

Thus, we get

$$E_Q[e^{-rT}(S_T - K)|\mathcal{F}_0] \leq S_0 + E_Q[G_T|\mathcal{F}_0] - e^{-rT}K$$
$$= S_0 - e^{-rT}K + E_Q[G_T|\mathcal{F}_0].$$

Thus we get the result. □

From this result, we see that if $E_Q[G_T|\mathcal{F}_0] < e^{-rT}K$, then for any martingale measures we get more small interval than those of above section 2.

4. Hedging options

We assume that discounted assets price process is martingale under a fixed measure Q equivalent to the given probability P in this section. Thus, we will omit the notation Q in the following. Consider the SDE (2.2) and the solution of it. We fix a finite horizon T and $k(t) := rt$. A trading strategy will be defined by an adapted process $\pi_t; 0 \leq t \leq T$ taking values in R and representing total amounts of assets held over time. We will constrain the processes $\pi_t^0; 0 \leq t \leq T$ and $\pi_t; 0 \leq t \leq T$ are left-continuous to take the jumps into account. The value process $V_t(\hat{\pi}); 0 \leq t \leq T$ with initial capital $x \in R_+$, corresponding to portfolio $\hat{\pi}_t := (\pi_t^0, \pi_t)$, is given by with initial condition $V_0 = x$;

$$V_t(\hat{\pi}) := \pi_t^0 e^{rt} + \pi_t \cdot S_t, \tag{4.1}$$

where $\cdot$ is a product. The strategy $\hat{\pi}_t$ is said to be self-financing if

$$dV_t(\hat{\pi}) = \pi_t^0 de^{rt} + \pi_t \cdot dS_t. \tag{4.2}$$

Equivalently, we get the condition of self-financing of strategy $\hat{\pi}_t$:

$$V_t(\hat{\pi}) = x + \int_0^t \pi_s^0 de^{rs} + \int_0^t \pi_s \cdot dS_s. \tag{4.3}$$

In the following, we denote value process only V_t. To make sense for this equation, we impose the condition:

$$\int_0^T |\pi_s^0| ds + \int_0^T |\pi_s|^2 ds < \infty \quad a.s.$$

For the future, we will impose a stronger condition of integrability on the processes $\pi_t; 0 \le t \le T$ as following: an admissible strategy is defined by an adapted, left-continuous process $\hat{\pi}_t; 0 \le t \le T$ with values in R^2 satisfying (4.3) $a.s.$ for all $t \in [0, T]$,

$$\int_0^T |\pi_s^0| ds < \infty, \quad P - a.s.$$

and

$$E[\int_0^T (\pi_s)^2 (S_s)^2 ds] < \infty. \tag{4.4}$$

PROPOSITION 4.1. *Let $\pi_t; 0 \le t \le T$ be an adapted, left continuous process such that the components $\pi_t; 0 \le t \le T$ hold (4.4). Let $V_0 \in R_+$. Then there exists a unique process $\pi_t^0; 0 \le t \le T$ such that $\hat{\pi}_t; 0 \le t \le T$ defines an admissible strategy with initial values V_0. Further, the discounted value process $\tilde{V}_t; 0 \le t \le T$ of strategy π_t is given by*

$$\tilde{V}_t = V_0 + \int_0^t \pi_s \cdot d\tilde{S}_s, \tag{4.5}$$

where $\tilde{S}_s = e^{-rs} S_s$ is a discounted asset price.

Proof. If $\hat{\pi}_t; 0 \le t \le T$ defines an admissible strategy, from (4.3), its value V_t at time t is given by $V_t = U_t + J_t$, where

$$U_t = V_0 + \int_0^t \pi_s^0 de^{rs} + \int_0^t \pi_s \cdot dS_c(s),$$
$$J_t = \int_0^t \pi_s \cdot dS_d(s).$$

Differentiating the products $e^{-rt}U_t$, $e^{-rt}J_t$ and $e^{-rt}V_t$, we get

$$e^{-rt}V_t - V_0 = \int_0^t -re^{-rs} U_s ds + \int_0^t e^{-rs} dU_s + \int_0^t -re^{-rs} J_s ds + \int_0^t e^{-rs} dJ_s.$$

Therefore we have that

$$\begin{aligned}\tilde{V}_t &= V_0 + \int_0^t -re^{-rs}(U_s + J_s)ds + \int_0^t e^{-rs} d(U_s + J_s) \\ &= V_0 + \int_0^t -re^{-rs} V_s ds + \int_0^t e^{-rs}\pi_s^0 de^{rs} + \int_0^t e^{-rs}\pi_s \cdot dS_s \\ &= V_0 - \int_0^t re^{-rs} V_s ds + \int_0^t r\pi_s^0 ds + \int_0^t e^{-rs}\pi_s \cdot dS_s.\end{aligned}$$

Then, by the definition (4.1), we get

$$\begin{aligned}\tilde{V}_t = V_0 &- \int_0^t re^{-rs}[\pi_s^0 e^{rs} + \pi_s \cdot S_s]ds \\ &+ \int_0^t r\pi_s^0 ds + \int_0^t e^{-rs}\pi_s \cdot dS_s \\ = V_0 &+ \int_0^t \pi_s e^{-rs} \cdot dS_s - \int_0^t r\pi_s \cdot \tilde{S}_s ds.\end{aligned}$$

But, from the fact that

$$d\tilde{S}_s = e^{-rs} dS_s - re^{-rs} S_s ds,$$

we get (4.5).

If V_0 and π_t are given, the unique process π_t^0 such that $\hat{\pi}_t; 0 \le t \le T$ is an admissible strategy with initial value V_0, is given by

$$\begin{aligned}\pi_t^0 &= \tilde{V}_t - \pi_t \cdot \tilde{S}_t \\ &= V_0 + \int_0^t \pi_s \cdot d\tilde{S}_s - \pi_t \cdot \tilde{S}_t\end{aligned}$$

by (4.1) and (4.5). From this formula, we see that the process π_t^0 is adapted, has left-hand limit at any point, and is such that $\pi_t^0 = \pi_{t-}^0$.□

From the condition (4.4), we get that

$$E[\int_0^T (\pi_s)^2 (\tilde{S}_s)^2 ds] < \infty.$$

From this condition, we see the discounted value process $\tilde{V}_t; 0 \leq t \leq T$ of an admissible strategy $\pi_t; 0 \leq t \leq T$ is again a square-integrable martingale.

Now, let us stand from the writer's point of view. We will deal for simple martingale model without Girsanov transformation, because it influence to the bounded variation part of semimartinglae only in our models. The writer sells the option at a price V_0 at time 0 and then follows an admissible strategy between times 0 and T. From Proposition 4.1, this strategy is completely determined by the process $\pi_t; 0 \leq t \leq T$ representing the amounts of the risky assets. If V_t represents the value of this strategy π_t at time t, the hedging mismatch at maturity is given by $h - V_T$. A way of measuring the risk is to introduce the quantity $R^T(0)$ from $R^T(t)$:

$$R^T(t) := E[(e^{-r(T-t)}(h - V_T))^2 | \mathcal{F}_t]. \tag{4.6}$$

We know that the minimal risk $R^T(0)$ at maturity T is following:

$$\begin{aligned} R^T(0) &= E[(e^{-rT}(g(S_T) - V_T))^2 | \mathcal{F}_0] \qquad (4.7) \\ &= E[(e^{-rT}(h - V_T))^2 | \mathcal{F}_0] \\ &= E[(\tilde{h} - \tilde{V}_T)^2 | \mathcal{F}_0], \end{aligned}$$

and the initial value of any admissible strategy aiming at minimizing the risk $R^T(0)$ at maturity T is given by

$$\begin{aligned} u(0, S_0) &:= E[e^{-rT} g(S_T) | \mathcal{F}_0] \qquad (4.8) \\ &= E[\tilde{h} | \mathcal{F}_0], \end{aligned}$$

where $\tilde{h} := e^{-rT} g(S_T)$ and g is some continuous payoff function of asset defined from $(0, \infty)(\subset R)$ to $[0, \infty)$.

Now, we determine a process $\pi_t; 0 \leq t \leq T$ which is a trading strategy for S_t of quantities of risky asset to be held in portfolio to minimize $R^T(0)$. To do so, we need the following proposition:

PROPOSITION 4.2. *Let V_t be the value of asset at time t of an admissible strategy with initial value $V_0 = E[e^{-rT}g(S_T)|\mathcal{F}_0]$, determined by a process $\pi_t; 0 \le t \le T$ for the quantities of risky asset. Then the quadratic risk at maturity $R^T(0)$ is given by*

$$R^T(0) = E[(\int_0^T \int_U \Phi(\tilde{S}_s, z)\tilde{N}_p(dz, ds))^2|\mathcal{F}_0], \tag{4.9}$$

where

$$\begin{aligned}\Phi(\tilde{S}_s, z) = \tilde{u}(s, \exp(zv)(\tilde{S}_{s-})) - \tilde{u}(s, \tilde{S}_s) \\ - \pi_s \cdot (\exp(zv)(\tilde{S}_{s-}) - \tilde{S}_{s-}).\end{aligned}$$

Proof. From the Proposition 4.1, we have that, for $t \le T$,

$$\begin{aligned}\tilde{V}_t &= V_0 + \int_0^t \pi_s \cdot d\tilde{S}_s \qquad (4.10)\\ &= u(0, \xi_0) - \int_0^t r\pi_s \cdot \tilde{S}_s ds + \int_0^t e^{-rs}\pi_s \cdot dS_s \\ &= u(0, \xi_0) - \int_0^t r\pi_s \cdot \tilde{S}_s ds + \int_0^t e^{-rs}\pi_s \\ &\quad \cdot [\int_U (e^z S_s - S_s - zS_s)\nu(dz)ds + \int_U (e^z S_{s-} - S_{s-})\tilde{N}_p(dz, ds) \\ &\quad + \int_{U^c} (e^z S_{s-} - S_{s-})N_p(dz, ds)],\end{aligned}$$

where dS_t is of the SDE form (3.2). We define function $\tilde{u}(t, x) := e^{-rt}u(t, xe^{rt})$ where $u(t, S_t) := E[e^{-r(T-t)}g(S_T)|\mathcal{F}_t]$. Then we get $\tilde{u}(t, \tilde{S}_t) = E[\tilde{h}|\mathcal{F}_t]$. It induce that $\tilde{u}(t, \tilde{S}_t)$ is the discounted price of option h at time t and that $\tilde{u}(t, \tilde{S}_t)$ is a martingale. From the definition of $u(t, x)$, we can deduce that $\tilde{u}(t, x)$ is $C^{1,2}$-function on $[0, T] \times R_+$. Thus from the Itô formula, we obtain for the S_t represented by (3.2) when $s = 0$ in solution (2.3);

$$\begin{aligned}\tilde{u}(t, \tilde{S}_t) = \tilde{u}(0, \tilde{S}_0) + \int_0^t \frac{\partial}{\partial s}\tilde{u}(s, \tilde{S}_s)ds \quad (4.11)\\ + \int_0^t \frac{\partial}{\partial x}\tilde{u}(s, \tilde{S}_s) \int_U (e^z \tilde{S}_s - \tilde{S}_s - z\tilde{S}_s)\nu(dz)ds\end{aligned}$$

$$+\int_0^t \int_U [\tilde{u}(s, \tilde{S}_{s-} + \Delta \tilde{S}_s) - \tilde{u}(s, \tilde{S}_s)$$
$$- \frac{\partial}{\partial x} \tilde{u}(s, \tilde{S}_s)(e^z \tilde{S}_s - \tilde{S}_s^i)] \hat{N}_p(dz, ds)$$
$$+\int_0^t \int_U [\tilde{u}(s, \tilde{S}_{s-} + \Delta \tilde{S}_s) - \tilde{u}(s, \tilde{S}_s)] \tilde{N}_p(dz, ds)$$
$$+\int_0^t \int_{U^c} [\tilde{u}(s, \tilde{S}_{s-} + \Delta \tilde{S}_s) - \tilde{u}(s, \tilde{S}_s)] N_p(dz, ds).$$

From the facts: $\tilde{S}_t = e^{-rt} S_t$, $S_t = e^{rt} \tilde{S}_t$, and $\Delta S_t = e^{rt} \Delta \tilde{S}_t$, we get

$$\Delta \tilde{S}_t = e^{-rt} \Delta S_t = e^{-rt} e^{rt} \Delta \tilde{S}_t = \tilde{S}_{t-}(e^{\Delta Z_t} - 1),$$

and

$$\tilde{S}_{t-} + \Delta \tilde{S}_t = \tilde{S}_{t-} + \tilde{S}_{t-}(e^{\Delta Z_t} - 1) = \tilde{S}_{t-} e^{\Delta Z_t}.$$

Thus we get that

$$\tilde{u}(t, \tilde{S}_t) = \tilde{u}(0, \tilde{S}_0) + \int_0^t \partial_s \tilde{u}(s, \tilde{S}_s) ds \quad (4.12)$$
$$+\int_0^t \int_U [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s) - \partial_x \tilde{u}(s, \tilde{S}_s) z \tilde{S}_s^i] \nu(dz) ds$$
$$+\int_0^t \int_U [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s)] \tilde{N}_p(dz, ds)$$
$$+\int_0^t \int_{U^c} [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s)] N_p(dz, ds).$$

Gathering equalities (4.10) and (4.12), from the fact:

$$\tilde{h} := e^{-rT} g(S_T) = e^{-rT} E[g(S_T) | \mathcal{F}_T]$$
$$= e^{-rT} u(T, \xi_T) = \tilde{u}(T, \tilde{S}_T),$$

we get that

$$\tilde{h} - \tilde{V}_T = \tilde{u}(T, \tilde{S}_T) - \tilde{V}_T \qquad (4.13)$$
$$= M^{(1)}(T) + M^{(2)}(T).$$

Here

$$
\begin{aligned}
M^{(1)}(T) &= \int_0^T \int_U [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s)] \tilde{N}_p(ds, dz) \\
&\quad - \int_0^T \pi_s \cdot \int_U (e^z \tilde{S}_{s-} - \tilde{S}_{s-}) \tilde{N}_p(ds, dz) \\
&= \int_0^T \int_U \Phi(\tilde{S}_s, z) \tilde{N}_p(ds, dz)
\end{aligned}
$$

is the martingale part, and the bounded variation part $M^{(2)}(T)$ of semimartingale is

$$
\begin{aligned}
M^{(2)}(T) &= \int_0^T \partial_s \tilde{u}(s, \tilde{S}_s) ds \\
&+ \int_0^T \int_U [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s) - \sum_{i=1}^d \partial_x \tilde{u}(s, \tilde{S}_s) z \tilde{S}^i(s)] \nu(dz) ds \\
&+ \int_0^T \int_{U^c} [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s)] N_p(ds, dz) \\
&+ \int_0^T r \pi_s^* \cdot \tilde{S}_s ds \\
&- \int_0^T \pi_s^* \cdot \int_U (e^z \tilde{S}_s - \tilde{S}_s - z \tilde{S}_s) \nu(dz) ds \\
&- \int_0^T \pi_s^* \cdot \int_{U^c} (e^z \tilde{S}_{s-} - \tilde{S}_{s-}) N_p(ds, dz).
\end{aligned}
$$

But from the fact that $\tilde{h} - \tilde{S}_T$ is again a martingale, we get that the bounded variation part $M^{(2)}(T)$ of semimartingale must be 0. Thus we get by (4.7) and (4.13):

$$
R^T(0) = E[(M^{(1)}(T))^2 | \mathcal{F}_0].
$$

Therefore, the risk at maturity $R^T(0)$ is got as (3.9). □

From this proposition, we get following result:

THEOREM 4.3. *The strategy π_t corresponding to the minimal risk $R^T(0)$ is given by*

$$\pi_t = D(t, \tilde{S}_t), \tag{4.14}$$

where

$$D(t,x) = \{\int_U (e^z x - x)^2 \nu(dz)\}^{-1} \cdot \{\int_U (e^z x - x)[\tilde{u}(s, e^z x) - \tilde{u}(t,x)]\nu(dz)\}.$$

Proof. From Proposition 4.2, we get that

$$\begin{aligned}
R^T(0) &= E[(\int_0^T \int_U \Phi(\tilde{S}_s, z)\tilde{N}_p(dz, ds))^2 | \mathcal{F}(0)] \qquad (4.15)\\
&= E[(\int_0^T \int_U \Phi(\tilde{S}_s, z) N_p(dz, ds) - \int_0^T \int_U \Phi(\tilde{S}_s, z)\nu(dz)ds)^2 | \mathcal{F}_0]\\
&= E[(\int_0^T \int_U \Phi(\tilde{S}_s, z)\tilde{N}_p(dz, ds))^2 - (\int_0^T \int_U \Phi(\tilde{S}_s, z)\nu(dz)ds))^2\\
&\quad + (\int_0^T \int_U \Phi(\tilde{S}_s, z)\nu(dz)ds)^2 | \mathcal{F}_0]\\
&= E[(\int_0^T \int_U \Phi(\tilde{S}_s, z)\nu(dz)ds)^2]\\
&= E[\int_0^T \int_U [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s) - \pi_s^* \cdot (e^z \tilde{S}_{s-} - \tilde{S}_{s-})]^2 \nu(dz)ds].
\end{aligned}$$

It follows that the minimal risk is obtained when π satisfies

$$\begin{aligned}
&\int_U [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s)\\
&\quad - \pi_s^* \cdot (e^z \tilde{S}_{s-} - \tilde{S}_{s-})](e^z \tilde{S}_{s-} - \tilde{S}_{s-})\nu(dz) = 0, \quad P - a.e..
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\int_U [\tilde{u}(s, e^z \tilde{S}_{s-}) - \tilde{u}(s, \tilde{S}_s)](e^z \tilde{S}_{s-} - \tilde{S}_{s-})\nu(dz)\\
&= \pi_s^* \cdot \int_U (e^z \tilde{S}_{s-} - \tilde{S}_{s-})^2 \nu(dz).
\end{aligned}$$

From this, we get the result. □

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