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A PINCHING THEOREM FOR RIEMANNIAN 4–MANIFOLD

Kwanseok Ko

Abstract. Let (M, g) be a compact oriented 4-dimensional Riemannian manifold whose sectional curvature k satisfies $1 \geq k \geq 0.1714$. We show that M is topologically S^4 or $\pm\mathbb{CP}^2$.

1. Introduction

The purpose of this paper is to prove the following .

Theorem A. *Let M be a smooth compact oriented Riemannian 4-manifold whose sectional curvature k satisfies $1 \geq k \geq 0.1714$. Then M is topologically a 4-sphere S^4 or a complex projective 2-plane $\pm\mathbb{CP}^2$.*

Seaman[4] proved that if the manifold M satisfies the pinching condition

$$1 \geq k \geq \frac{1}{3\sqrt{1+\frac{3\cdot 2^{1/4}}{5^{1/2}}}+1} \approx 0.1714,$$

then M is definite.

Under this condition , we obtained the inequality

$$|\tau(M)| < \frac{1}{2}\chi(M), \tag{1}$$

where $\chi(M)$ the Euler characteristic of M and $\tau(M)$ is the signature of M . The idea of proof of the inequality (1) was originally due to Ville[8]. It follows easily that the second Betti number satisfies $b_2(M) \leq 1$. Since M is compact, even dimensional, oriented, and positively curved,

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it is simply connected by Synge theorem[6]. Freedman's classification theorem[2] states that smooth compact simply connected 4-manifolds are classified topologically by their intersection form. Therefore M is topologically a 4-sphere S^4 or a complex projective 2-plane $\pm\mathbb{CP}^2$. This result was announced in Ko[3]. By adapting Ville's argument , we get the inequality(1). The we have the conclusion of the theorem.

2. Estimates of curvature tensor

In this section, we introduce Ville's method and include her's lemmas and proofs for the completeness of the theorem.

Let M be an oriented 4- manifolds and let $T_p(M)$ be the tangent space of M at a fixed point $p \in M$. The rank-6 bundle of 2-forms Λ^2 on an oriented Riemannian 4-manifold (M^4, g) has an invariant decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{2}$$

as the sum of two rank-3 vector bundles. Here $\Lambda^\pm$ are by definition the eigenspaces of the Hodge $*$ operator

$$\star : \Lambda^2 \to \Lambda^2,$$

corresponding respectively to the eigenvalue ± 1. Sections of Λ^+ are called *self-dual* 2-forms, whereas sections of Λ^- are called *anti-self-dual* 2-forms.

Let $\tilde{G}$ be the subspace of Λ^2 consisting of unitary simple bivectors (that is of $x \wedge y$ where x and y are unitary and orthogonal). Let $G = \tilde{G}/\pm 1$ be the 2-dimensional Grassmannia manifold of $T_p(M)$.

Lemma 1. *Let $H \in \Lambda^+$ and $K \in \Lambda^-$.*
Then

$$\frac{H+K}{\sqrt{2}} \in \tilde{G} \Longleftrightarrow ||H|| = ||K|| = 1.$$

We can also make use of this splitting of Λ^2 to write the matrix of curvature tensor R [5]:

$$R = \left(\begin{array}{c|c} W^+ + uId_{\Lambda^+} & Z_1 \\ \hline Z_1 & W^- + uId_{\Lambda^-} \end{array} \right), \tag{3}$$

where $U = uId_{\Lambda^2}$ and W^+ and W^- are the matrix of *self-dual* and *anti-self-dual Weyl curvatures* respectively. The matrix of Weyl curvature $W = W^+ + W^-$ is trace-free .

$$Z = \left(\begin{array}{c|c} 0 & Z_1 \\ \hline Z_1 & 0 \end{array} \right),$$

represents the matrix of trace-free Ricci curvature tensor.

The curvatures $W^{\pm}$, Z, and U correspond to different irreducible representation of $SO(4)$, so the only invariant quadratic polynomials in the curvature of an oriented 4-manifold are linear combinations of $||U||^2$, $||Z||^2$, $||W^+||^2$ and $||W^-||^2$. This observation can be applied, in particular, to simplify the integrands [1] of the 4-dimensional Chern-Gauss-Bonnet

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left[||U||^2 + ||W^+||^2 + ||W^-||^2 - ||Z||^2 \right] d\mu \tag{4}$$

and Hirzebruch signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left[||W^+||^2 - ||W^-||^2 \right] d\mu \tag{5}$$

formulæ. Here the curvatures, norms $|| \cdot ||$, and volume form $d\mu$ are, of course, those of any given Riemannian metric g on M.

Let us assume that $\int_M \left[||W^+||^2 - ||W^-||^2 \right] d\mu \geq 0$ (possibly by changing the orientation of M : our purpose will then be to give a lower estimate for:

$$\Delta(R) = ||U||^2 - \frac{1}{3}||W^+||^2 + \frac{7}{3}||W^-||^2 - ||Z||^2. \tag{6}$$

The pinching hypothesis yields the following.

In the followings. we let $\delta = 0.1714$.

LEMMA 2.

$$(a)\forall P \in \tilde{G}, \delta \leq < (U+W)P, P > \leq 1.$$

$$(b)\forall H \in \Lambda^+, \delta \leq u + \frac{1}{2} < W^+H, H > \leq 1.$$

Proof. (a) $< (U+W)P, P >= \frac{1}{2}[< RP, P > + < R * P, P >]$.
(b) The matrix of the quadratic from definition Λ^- by

$$K \mapsto < W^-K, K >$$

is of trace zero, hence it admits a unitary isotropic vector K_0. Let us consider

$$P = \frac{H + K_0}{\sqrt{2}} \in \tilde{G}.$$

We get :

$$< (U+W)P, P >= u + \frac{1}{2} < W^+H, H > .$$

□

Now we estimate the various curvature components of the equation separately.

W^+ is a symmetric mapping of Λ^+, hence Λ^+ possesses an orthonormal basis of eigenvectors $\{H_1, H_2, H_3\}$.
(a) Let $W^+H_i = w_i^+ H_i$: w_i^+'s are the eigenvalues of W^+.
(b) Let $z_i \in \underset{\sim}{R}, K_i \in \Lambda^-, i = 1, 2, 3$, be such that

$$||K_i|| = 1,$$
$$ZH_i = z_i K_i.$$

We get

$$z_i^2 =< ZH_i, K_i >^2 = ||ZH_i||^2.$$

(c) Let $w_i^- =< W^-K_i, K_i >$.

Then the w_i^- 's are not eigenvalues of W^-.

(d) Let

$$v_i = u + \frac{w_i^+}{2}$$
$$=< (U + W^+), (\frac{H_i + K_0}{\sqrt{2}}), (\frac{H_i + K_0}{\sqrt{2}}) > .$$

LEMMA 3.

$$||Z||^2 \leq 2 \sum_{i=1}^{3} A_i^2,$$

where,

$$A_i = \min[1 - v_i + \frac{w_i^-}{2}, v_i + \frac{w_i^-}{2} - \delta].$$

Proof. According to lemma2, the $P_i^{\pm} = \frac{H_i + K_i}{\sqrt{2}}$ belong to $\tilde{G}$, the pinching hypothesis yields

$$\delta \leq v_i + \frac{w^-}{2} \pm < ZH_iK_i > \leq 1$$

and hence

$$||Z||^2 = 2 \sum_{i=1}^{3} ||ZH_i||^2 \leq \sum_{i=1}^{3} A_i^2.$$

□

We now compute

$$||U||^2 - \frac{1}{3}||W^+||^2 = 6u^2 - \frac{1}{3}\sum_{i=1}^{3}(w_i^+)^2$$
$$= -\frac{4}{3}\sum_{i=1}^{3} v_i^2 + \frac{10}{9}(\sum_{i=1}^{3} v_i)^2.$$

If we put $\alpha = \sup_{1,2,3} |w_i^-|$, then using the fact $tr(W^-) = 0$ we have a lower bound for $||W^-||^2$:

$$||W^-||^2 \geq \frac{3}{2}\alpha^2.$$

.

We can now derive from the preceding estimates

(7)
$$\frac{\Delta}{2} \geq \frac{5}{9}(\sum_{i=1}^{3} v_i)^2 - \frac{2}{3}\sum_{i=1}^{3} v_i^2 + \frac{7}{4}\alpha^2 - \sum_{i=1}^{3} min[(1-v_i+\frac{w_i^-}{2})^2, (v_i+\frac{w_i^-}{2}-\delta)^2].$$

3. The proof of Main Theorem

Let us define 2 real valued functions, m and H of respectively 1 and 3 real variables:

$$m(x) = \min(1-x, x-\delta),$$

(8)
$$H(x_1, x_2, x_3) = \frac{9}{5}(\sum_{i=1}^{3} x_i)^2 - \frac{2}{3}\sum_{i=1}^{3} x_i^2 - \sum_{i=1}^{3} m(v_i)^2 - \frac{1}{5}(\sum_{i=1}^{3} m(x_i))^2.$$

We transform (7) by making use of

$$\frac{7}{4}\alpha^2 - \sum_{i=1}^{3} min[(1-x_i+\frac{w_i^-}{2})^2, (x_i+\frac{w_i^-}{2}-\delta)^2] \geq -\sum_{i=1}^{3} m(x_i)^2 - \frac{1}{5}(\sum_{i=1}^{3} m(x_i))^2.$$

This last inequality, together with (7) yield

$$\frac{\Delta}{2} \geq H(v_1, v_2, v_3).$$

According to lemma2 we just need to show that H is positive on

$$E = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \delta \leq x_3 \leq x_2 \leq x_1 \leq 1\}.$$

We split E into the union of 4 convex subdomains E_i , separated by the hyperplanes $x_i = \frac{1+\delta}{2}$: on each of the E_i's, H turns out to be concave. We check its values on the extreme points of the E_i's: they are all positive. The proof of inequality (1) is thus complete.

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Department of Mathematics
Inha University
Incheon, 402-751, Korea
E-mail: ksko@math.inha.ac.kr