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STUDY ON THE JOINT SPECTRUM

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Abstract. We introduce the Joint spectrum on the complex Banach space and on the complex Hilbert space and the tensor product spectrums on the tensor product spaces. And we will show $\sigma[P(T_1, T_2, \ldots, T_n)] = \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n)$ on $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ for a polynomial P.

1. Introduction

Let $BL(X)$ denote the algebra of bounded linear operator acting on the complex Banach space X. If $T \in BL(X)$, then write $N(T)$ and $R(T)$ for the null space and the range of T ; $\alpha(T) = \dim(N(T))$; $\beta(T) = \text{codim}(R(T))$; $\sigma(T)$ for the spectrum of T .

An operator $T \in BL(X)$ is called Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension.

The index of a Fredholm operator $T \in BL(X)$ is givin by $i(T) = \alpha(T) - \beta(T)$.

An operator $T \in BL(X)$ is called Weyl if it is Fredholm of index zero.

An operator $T \in BL(X)$ is called Browder if it is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$.

The essential spectrum $\sigma_e(T)$, the Weyl's spectrum $w(T)$, and the Browder's spectrum $\sigma_b(T)$ are defined by

$$\begin{aligned} \sigma_e(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not Fredholm}\}; \\ w(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not Weyl}\}; \\ \sigma_b(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not Browder}\}; \\ \sigma(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not invertible}\}. \end{aligned}$$

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Then $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup acc(\sigma(T)) \subseteq \sigma(T)$ ([15]) , where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.

Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of commuting operators(bounded linear operators) on a complex Banach space X.

Then the joint spectrum of $T = (T_1, T_2, \ldots, T_n)$ with respect to $BL(X)$ is to be the set of n-tuples $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers for which the system $T - \lambda = (T_1 - \lambda_1, T_2 - \lambda_2, \ldots, T_n - \lambda_n)$ generates a proper left or right ideal in $BL(X)$. The joint spectrum of T is denoted by $\sigma(T) = \sigma^l(T) \cup \sigma^r(T)$, where $\sigma^l(T) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | I \notin \sum_i BL(X)(T_i - \lambda_i)\}$ and $\sigma^r(T) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | I \notin \sum_i (T_i - \lambda_i) BL(X), 1 \leq i \leq n\ \}$ ([9]).

Let $T = (T_1, T_2, \ldots, T_n)$ be n-tuples of commuting bounded linear operators defined on a complex Banach space X and $T_i \in BL(X), i = 1, 2, \ldots, n$. Define $M_k(T) = R(T_1^k) + R(T_2^k) + \cdots + R(T_n^k)$ for $k \in N$, where $T(x) = (T_1, T_2, \ldots, T_n)(x) = T_1(x) + T_2(x) + \cdots + T_n(x)$ for each $x \in X$.Clearly, $X = M_0(T) \supseteq M_1(T) \supseteq \ldots$ holds. Set $R^\infty(T) = \cap_{k=1}^\infty M_k(T)$. We say that $T = (T_1, T_2, \ldots, T_n)$ is lower semi-Fredholm, that is; $T \in \Phi -^{(n)} (X)$, if $\operatorname{codim}(M_1(T)) = \operatorname{codim}(R(T_1) + R(T_2) + \cdots + R(T_n)) < \infty$.$T = (T_1, T_2, \ldots, T_n)$ is lower semi-Weyl, that is; $T \in W -^{(n)} (X)$, if $\operatorname{codim}(R^n(T)) = \operatorname{codim}(\cap_{k=1}^n M_k(T)) < \infty$ for all $n \geq 2$. And $T = (T_1, T_2, \ldots, T_n)$ is lower semi-Browder, that is; $T \in B -^{(n)} (X)$, if $\operatorname{codim}(R^\infty(T)) = \operatorname{codim}(\cap_{k=1}^\infty M_k(T)) < \infty$. Since $\operatorname{codim}(M_1(T)) < \infty$ implies $\operatorname{codim}(M_k(T)) < \infty$ for every $k \in N$ ([13]).

We have the inclusion $\Phi -^{(n)} (X) \subseteq W -^{(n)} (X) \subseteq B -^{(n)} (X)$.

The lower semi-Fredholm spectrum $\sigma_\Phi - (T)$; the lower simi-Weyl spectrum $\sigma_W - (T)$; the lower semi-Browder spectrum $\sigma_B - (T)$ and the defect spectrum $\sigma_{\delta(T)}$ of $T = (T_1, T_2, \ldots, T_n)$ are defined by

$$\begin{aligned}
\sigma_\Phi - (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin \Phi -^{(n)} (X)\}; \\
\sigma_W - (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin W -^{(n)} (X)\}; \\
\sigma_B - (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin B -^{(n)} (X)\}; \\
\sigma_\delta - (T) &= \{\lambda \in \mathbb{C}^n | \operatorname{codim}(M_1(T - \lambda I)) \neq 0\} \qquad ([1],[3]).
\end{aligned}$$

We say that $T = (T_1, T_2, \ldots, T_n)$ is upper semi-Fredholm, that is; $T \in \Phi +^{(n)} (X)$, if the map $T : X \to X^n$ defined by $T(x) = (T_1(x), \ldots, T_n(x))$ is upper semi-Fredholm; equivalently, if T has finite dimensional null space and closed range;, $T = (T_1, T_2, \ldots, T_n)$ is upper semi-Weyl, that is, $T \in W +^{(n)} (X)$, if $T \in \Phi +^{(n)} (X)$ and $\dim(N^n(T)) = \dim(\cup_{k=1}^n [N(T_1^k) \cap$

$N(T_2^k) \cap \cdots \cap N(T_n^k)]) < \infty$ for all $n \geq 2$, and $T = (T_1, T_2, \ldots, T_n)$ is upper semi-Browder, that is, $T \in B +^{(n)} (X)$, if $T \in \Phi +^{(n)} (X)$ and $\dim(N^\infty(T)) = \dim(\cup_{k=1}^{\infty}[N(T_1^k) \cap N(T_2^k) \cap \cdots \cap N(T_n^k)]) < \infty$.

The upper semi-Fredholm spectrum $\sigma_\Phi + (T)$, the uppper semi-Weyl spectrum $\sigma_w + (T)$, the upper semi-Browder spectrum $\sigma_B + (T)$, and the approximate point spectrum $\sigma_\pi(T)$ of $T = (T_1, T_2, \ldots, T_n)$ are defined by([1], [13]);

$$\begin{aligned}
\sigma_\Phi + (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin \Phi +^{(n)} (X)\}; \\
\sigma_W + (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin W +^{(n)} (X)\}; \\
\sigma_B + (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin B +^{(n)} (X)\}; \\
\sigma\pi + (T) &= \{\lambda \in \mathbb{C}^n | N(T - \lambda I) \neq 0 \text{ or } R(T - \lambda I) \text{is not closed}\}.
\end{aligned}$$

Let $T = (T_1, T_2, \ldots, T_n)$ be a commuting n-tuple of bounded linear operators defined on a complex Banach space X and let $C - (T) = \{\lambda \in \mathbb{C}^n | N(T - \lambda I)$ has not a direct complement in X^n, where a map $T : X^n \to X$ defined by $T(x_1, x_2, \ldots, x_n) = T_1(x_1) + T_2(x_2) + \cdots + T_n(x_n)\}$.

We say that $SP_e - (T) = \sigma_\Phi - (T) \cup C - (T)$ is the lower split semi-Fredholm spectrum of T, $SR_W - (T) = \sigma_w - (T) \cup C - (T)$ is the lower split semi-Weyl spectrum of T, $SP_B - (T) = \sigma_B - (T) \cup C - (T)$ is the lower split semi-browder spectrum of T, and $SP_\delta(T) = \sigma_\delta(T) \cup C - (T)$ is the split defect spectrum of T.

Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of commuting bounded linear operators defined on a complex Banach space X.

Define the map $T : X \to X^n$ by $T(x) = (T_1(x), T_2(x), \cdots, T_n(x))$. Let $C + (T) = \{\lambda \in \mathbb{C}^n | R(T - \lambda I)$ has not a direct complement in $X^n\}$.

We say that $SP_e + (T) = \sigma_\Phi + (T) \cup C + (T)$ is the upper split semi-Fredholm spectrum of T, $SP_W + (T) = \sigma_W + (T) \cup C + (T)$ is the upper split semi-Weyl spectrum of T, $SP_B + (T) = \sigma_B + (T) \cup C + (T)$ is the upper split semi-Browder spectrum of T, and $SP_\pi(T) = \sigma_\pi(T) \cup C + (T)$ is the split approximate point spectrum of T.

Let $X_1, \ldots, X_n$ be the complex Banach spaces and $X = X_1 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some uniform, reasonable cross-norm([5], [11]). Let I_k be the identity operator on X_k and A_k an arbitrary bounded linear operator on $X_k, 1 \leq$

$k \leq n$.

$$\begin{aligned} Set \quad T_1 &= A_1 \otimes I_2 \otimes \cdots \otimes I_n \\ &\vdots \\ T_n &= I_1 \otimes I_2 \otimes \cdots \otimes A_n. \end{aligned}$$

By ([4], [6]), $\sigma(T_k) = \sigma(A_k), 1 \leq k \leq n$.

We call $\sigma(T_1, T_2, \ldots, T_n) = \prod_{k=1}^{n} \sigma(T_k) = \prod_{k=1}^{n} \sigma(A_k) = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_k \in \sigma(T_k), 1 \leq k \leq n\}$ the Joint spectrum of $T = (T_1, T_2, \ldots, T_n)$ on $X_1 \otimes X_2 \otimes \cdots \otimes X_n$. If $\sigma(T, X)$ is the Joint spectrum of $T = (T_1, T_2, \ldots, T_n)$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$, then by ([6, theorem 1]), we have
$\sigma(T, X) = \prod_{k=1}^{n} \sigma(A_k), 1 \leq k \leq n.$

2. Main result

THEOREM 2.1. *Let $T = (T_1, T_2, \ldots, T_2)$ be a commuting n-tuple of bounded linear operators defined on a complex Banach space X. Then the following statements hold:*

(1) $\sigma_\Phi - (T) \subseteq \sigma_W - (T) \subseteq \sigma_B - (T) \subseteq \sigma_\delta(T)$;
(2) $\sigma_\Phi + (T) \subseteq \sigma_W + (T) \subseteq \sigma_B + (T) \subseteq \sigma_\pi(T)$;
(3) $\sigma_\Phi - (T) \subseteq SP_e - (T) \subseteq SP_W - (T) \subseteq SP_B - (T) \subseteq SP_\pi(T)$;
(4) $\sigma_\Phi + (T) \subseteq SP_e + (T) \subseteq SP_W + (T) \subseteq SP_B + (T) \subseteq SP_\pi(T)$.

Proof. Since $\text{codim}(M_1(T)) < \infty$ implies $\text{codim}(M_k(T)) < \infty$ for every $k \in N$ and $\text{codim}(R^n(T)) = \text{codim}(\cap_{k=1}^{n} M_k(T)) < \infty$ for all $n \geq 2$,

$$\text{codim}(R^\infty(T)) = \text{codim}(\cap_{k=1}^{\infty}(M_k(T)) < \infty \qquad ([13], [1])$$

Easy calculations show that (1) and (2) hold. Since $SP_e - (T) = \sigma_\Phi - (T) \cup C - (T), SP_e + (T) = \sigma_\Phi + (T) \cap C + (T), \sigma_B - (T) \subseteq \sigma_\delta(T)$ and $\sigma_B + (T) \subseteq \sigma_\pi(T)$ ([1]), and by [12], it is easy to see that (3) and (4) hold. □

We now see that the joint spectrum we have introduced satisfy the main spectral properties.

THEOREM 2.2. *Let $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ be a tensor product of the complex Hilbert space $X_i, 1 \leq i \leq n$. Let $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to* *some cross norm and let A_i be a bounded linear operator on $X_i, 1 \leq i \leq n$. Suppose that T_i is the operator on $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ defined by $T_i = I_1 \otimes I_2 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n$ and $T_n = I_1 \otimes I_2 \otimes \cdots \otimes I_{n-1} \otimes A_n$, where I_i is the identity operator on $X_i, 1 \leq i \leq n$. Then, $\sigma_\Phi(T_1, T_2, \cdots, T_n) \subseteq \sigma_W(T_1, T_2, \ldots, T_n) \subseteq \sigma_B(T_1, T_2, \ldots, T_n) \subseteq \sigma(T_1, T_2, \ldots, T_n)$.*

Proof. Since the operators T_i obviously commute, we have $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n$ ([4], [6]). A complex vector $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(T_1, T_2, \ldots, T_n)$ *if and only if* $\lambda_i \in \sigma(A_i), 1 \leq i \leq n$, that is, $\sigma(T_1, T_2, \ldots, T_n) = \prod_{i=1}^n \sigma(A_i)$ [7,2.1 theorem], [4, Theorem 1]). And so we can verify the following results:
$\sigma_\Phi(T_1, T_2, \ldots, T_n) = \prod_{i=1}^n \sigma_e(A_i) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_i \in \sigma_e(A_i), 1 \leq i \leq n\}$
$\sigma_W(T_1, T_2, \ldots, T_n) = \prod_{i=1}^n \sigma_W(A_i) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_i \in \sigma_W(A_i), 1 \leq i \leq n\}$ and $\sigma_B(T_1, T_2, \ldots, T_n) = \prod_{i=1}^n \sigma_b(A_i) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_i \in \sigma_b(A_i), 1 \leq i \leq n\}$.
Since $\sigma_e(T) \subseteq W(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$, we have $\sigma_\Phi(T_1, T_2, \ldots, T_n) \subseteq \sigma_W(T_1, T_2, \ldots, T_n) \subseteq \sigma_B(T_1, T_2, \ldots, T_n) \subseteq \sigma(T_1, T_2, \ldots, T_n)$. □

Let $T_1, T_2, \ldots, T_n$ be a bounded linear operators on a Hilbert space X and let $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ be a tensor product on space $X_1 \otimes X_2 \otimes \cdots \otimes X_n$. Then $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n) = \{\lambda \in \mathbb{C} | \lambda = \lambda_1 \lambda_2 \cdots \lambda_n, \lambda_i \in \sigma(T_i), 1 \leq i \leq n\}$ ([3]).

We see that the tensor product operator we have introduced satisfy the following spectral properties.

Theorem 2.3. *Let $T_1, T_2, \ldots, T_n$ be a bounded linear operators on a complex Hilbert space. Then, $\sigma_\Phi(T_1 \otimes T_2 \otimes \cdots \otimes T_n) \subseteq \sigma_W(T_1 \otimes T_2 \otimes \cdots \otimes T_n) \subseteq \sigma_B(T_1 \otimes T_2 \otimes \cdots \otimes T_n) \subseteq \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n)$.*

Proof. By [3, 95-96], We have that $\sigma_\Phi(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma_e(T_1)\sigma_e(T_2)$ $\sigma_e(T_3) \cdots \sigma_e(T_n) = \{\lambda = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in \sigma_e(T_i),\ 1 \leq i \leq n\}, \sigma_W(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = w(T_1)w(T_2) \cdots w(T_n) = \{\lambda = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in w(T_i), 1 \leq i \leq n\}$, and $\sigma_B(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma_b(T_1)\sigma_b(T_2) \cdots \sigma_b(T_n) = \{\lambda = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in \sigma_b(T_i), 1 \leq i \leq n\}$.
Since $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$, we obtain the desired result. □
Let $X_1, X_2, \ldots, X_n$ be complex Banach space and let $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm.

Let T_i be the operator on $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ defined by $T_i = I_1 \otimes I_2 \otimes \cdots \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n$ for $A_i \in BL(X_i), 1 \leq i \leq n$. Since the operators T_i obviously commute, $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n$([4], [6]). Let $P(z_1, z_2, \ldots, z_n)$ be a polynomial in n variables. Then we can form the operator $P(T_1, T_2, \ldots, T_n)$([2]).

THEOREM 2.4. *Let $X_1, X_2, \ldots, X_n$ be complex Banach spaces and let $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm. If T_i is the operator on $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ defined by $T_i = I_1 \otimes I_2 \otimes \cdots \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n$ for $A_i \in BL(X_i), 1 \leq i \leq n$, then*

$$\sigma_\Phi\,[P(T_1, T_2, \ldots, T_n)] \subseteq \sigma_W[P(T_1, T_2, \ldots, T_n)] \subseteq$$
$$\sigma_B\,[P(T_1, T_2, \ldots, T_n)] \subseteq \sigma[P(T_1, T_2, \ldots, T_n)].$$

Proof. Martin Schechter show that $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n$ and $\sigma[P(T_1, T_2, \ldots, T_n)] = P[\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n)] = \{P(\lambda_1, \lambda_2, \ldots, \lambda_n) | \lambda_i \in \sigma(T_i) = \sigma(A_i), 1 \leq i \leq n\}$ ([2, Theorem 2.1], [9, Proposition 1.2]). Then we have
$\sigma_\Phi[p(T_1, T_2, \ldots, T_n)] = P(\sigma_e(T_1), \sigma_e(T_2), \ldots, \sigma(T_n))\{P(\lambda_1, \lambda_2, \ldots, \lambda_n)$
$|\lambda_i \in \sigma_e(T_i) = \sigma_e(A_i), 1 \leq i \leq n\}$,
$\sigma_w[P(T_1, T_2, \ldots, T_n)] = P(w(T_1), w(T_2), \ldots, w(T_n)) = \{P(\lambda_1, \lambda_2, \ldots, \lambda_n) | \lambda_i \in w(T_i) = w(A_i), 1 \leq i \leq n\}$, *and*
$\sigma_B[P(T_1, T_2, \ldots, T_n)] = P(\sigma_b(T_1), \sigma_b(T_2), \ldots, \sigma_b(T_n)) = \{P(\lambda_1, \lambda_2, \ldots, \lambda_n)$
$|\lambda_i \in \sigma_b(T_i) = \sigma_b(A_i), 1 \leq i \leq n\}$.

Since $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$ we obtain the desired result. □

Let $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a polynomial on n variables such that $P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C}$ for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$.
Then we can obtain the following result.

THEOREM 2.5. *Let X_i be a complex Banach space and $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm. Let A_i be a bounded linear operators on $X_i, 1 \leq i \leq n$ and let T_i be the operators on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ defined by $T_1 = A_1 \otimes I_2 \otimes I_3 \otimes \cdots \otimes I_n$, and, in general, $T_i = I_1 \otimes I_2 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n, 1 \leq i \leq n$ where I_i is the identity operator on $X_i, 1 \leq i \leq n$.*
Suppose that $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a polynomial in n variables such that

$P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1\lambda_2 \cdots \lambda_n$ for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$.
Then $\sigma[P(T_1, T_2, \ldots, T_n)] = \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n)$.

Proof. I_n[3], Brown and Pearcy show that $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n)$, where $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ is the tensor product of $T_1, T_2, \ldots, T_n$ acting on a Hilbert space $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$. But in [2], Martin Schechter show that for a complex Banach space X.

$$\begin{aligned} \sigma[P(T_1, T_2, \ldots, T_n)] &= P[\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n)] \\ &= \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n). \end{aligned}$$

In[6, Corollary 3], B.P. Rynne shows that the spectrum of the operator $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ is given by $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \{\lambda_1\lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in \sigma(T_i), 1 \leq i \leq n\}$. That is, $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n)$.
In[2], Martin Schechter show that $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n$. We obtain the desired result. □

Let $B_1, B_2, \ldots$, and B_n be subsets of $\mathbb{C}$ and let $P(B_1, B_2, \ldots, B_n)$ be a polynomial in n variables such that $P(B_1, B_2, \ldots, B_n) = B_1 \times B_2 \times \cdots \times B_n = \{z = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n | a_i \in B_i, 1 \leq i \leq n\}$.
Let us state the following result.

THEOREM 2.6. *Let $X_1, X_2, \ldots$, and X_n be complex Banach spaces and let A_k be bounded linear operators on $X_k, 1 \leq k \leq n$. Let $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some uniform, reasonable crossnorm, and let $T_k = I_1 \otimes I_2 \otimes \cdots \otimes A_k \otimes I_{k+1} \otimes \cdots \otimes I_n$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n, 1 \leq k \leq n$, where I_k is the identity operator on X_k. Then $\sigma[P(T_1, T_2, \ldots, T_n)] = \sigma[(T_1, T_2, \ldots, T_n)] = \sigma[(A_1, A_2, \ldots, A_n)]$.*

Proof. In[2] Martin Schechter show that $\sigma[P(T_1, T_2, \ldots, T_n)] = P(\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n))$.
And by Definition of $P(B_1, B_2, \ldots, B_n)$,

$$\begin{aligned} P[\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n)] &= \sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_n) \\ &= \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_k \in \sigma(T_k), 1 \leq k \leq n\}. \end{aligned}$$

On the other hand, in[6] B.P. Rynne show that $\sigma[(T_1, T_2, \ldots, T_n)] = \sigma(T_1) \times \sigma(T_2) \times \cdots \times \sigma(T_n)$. In[2], Martin Schechter show that $\sigma(T_k) = \sigma(A_k), 1 \leq k \leq n$. We obtain the desired result. □

References

[1] Enrico Boasso, Tensor Products and the semi-Browder Joint Spectra, J. Operator Theory 47(2002), 79-89.

[2] Martin Schechter, On the Spectra of Operators on Tensor Products. Journal of Functional Analysis 4(1969), 95-99.

[3] Arlen Brown and Carl Pearcy, Spectra of Tensor Products of Operators. Proc. Amer. Math. Soc. 17(1966), 162-165.

[4] A.T. Dash and M. Schechter, Tensor products and Joint spectra. Israel J. Math. Vol. 8(1970), 191-193.

[5] Volker Wrobel, Tensor Products of Linear Operators in Banach spaces and Taylors' Joint Spectrum. Operator Theory16(1986), 273-275.

[6] B. P. Rynne, Tensor Products and Taylors' Joint Spectrum in Banach Spaces. R. Ir. Acad. Vol.87A, No.1,(1987), 83-88.

[7] Zoia Ceausescu and F. H. Vasilescu, Tensor Products and Taylors' Joint Spectrum. Studia Mathematica, T. LXII.(1978), 305-310.

[8] Takashi Ichinose, Spectra Properties of Tensor Products of Linear Operator. I. Transactions of the America Mathematical Society Volume 235, (1978), 75-81.

[9] Robin Harte, The Spectral Mapping Theorem in Several Variables. Bulletin of the American Mathematical Society Volume 78, 5(1972), 871-872.

[10] Jörg Eschmeier at Münster, Tensor Products and Elementary Operators. J. Reine Angew. Math. 390(1988), 49-58.

[11] A.T. Dash, Joint Browder Spectra and Tensor Products. Bull. Austral. Math. Soc. Vol. 32(1985), 119-126.

[12] Vladimir Kordula, Vladimir Müller and Vladimir Rakocevic, On the semi-Browder Spectrum. Studia Mathematica 123(I), (1997), I-8.

[13] Josep L.Taylor, A Joint Spectrum for several Commuting Operators, Journal of Functional Analysis, 6 (1970), 172-173.

[14] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Dekker, New York, 1988, 493-525.

[15] L.A. Coburn and Martin Schechter, Joint Spectra and Interpolation of Operators, Journal of Functional Analysis 2, (1968), 226-237.

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