SELF-DUAL EINSTEIN MANIFOLDS OF POSITIVE
SECTIONAL CURVATURE

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ABSTRACT. Let \((M, g)\) be a compact oriented self-dual 4-
dimensional Einstein manifold with positive sectional curvature.
Then we show that, up to rescaling and isometry, \((M, g)\) is \(S^4\) or 
\(\mathbb{CP}^2\), with their canonical metrics.

1. Introduction and preliminaries

Let \((M, g)\) be an oriented Riemannian 4-manifold, and let \(\Lambda^2\) denote
the bundles of exterior 2-forms on \(M\). We have an invariant decomposition
\[
\Lambda^2 = \Lambda^+ \oplus \Lambda^- 
\]
as the sum of two vector bundles. Here \(\Lambda^\pm\) are the eigenspaces of the
Hodge star operator
\[
\star : \Lambda^2 \to \Lambda^2,
\]
corresponding respectively to the eigenvalue \(\pm 1\). Sections of \(\Lambda^+\) are
called \textit{self-dual} 2-forms, whereas sections of \(\Lambda^-\) are called \textit{anti-self-dual}
2-forms. But since the curvature tensor of \(g\) may be thought of as a
symmetric map \(\mathcal{R} : \Lambda^2 \to \Lambda^2\) given by
\[
\mathcal{R}(e_{ij}) = \frac{1}{2} \sum_{k,l} R_{ijlk} e_{kl},
\]
where \(\{e_i\}\) is a local orthonormal basis of 1-forms, \(e_{ij}\) denotes the 2-
form \(e_i \wedge e_j\) and \(R_{ijlk} = \langle R(e_i, e_j) e_l, e_k \rangle\). Equation (1) gives us a
decomposition [1, 9] of the curvature into irreducible components

\[
R = \begin{pmatrix}
W^+ + \frac{s}{12} & Z \\
Z & W^- + \frac{s}{12}
\end{pmatrix},
\]

where the self-dual and anti-self-dual Weyl curvatures \( W^\pm \) of Weyl curvature \( W = W^+ + W^- \) are trace-free as endomorphisms of \( \Lambda^\pm \). The scalar curvature \( s \) is understood here to act by scalar multiplication. On the other hand, \( Z \) represents the trace-free Ricci curvature \( \text{Ric} - \frac{s}{4}g \).

Let \( A = W^+ + \frac{s}{12} \) and let \( C = W^- + \frac{s}{12} \). They are \( trA = trC = \frac{s}{4} \). Then the two components of the Weyl tensor \( W^+ \) and \( W^- \) are given by \( W^+ = A - \frac{s}{12} \) and \( W^- = C - \frac{s}{12} \).

Riemannian 4-manifold \((M, g)\) is said to be \textit{Einstein} if it has constant Ricci curvature — i.e. if its Ricci tensor \( \text{Ric} \) is a constant multiple of the metric:

\[
\text{Ric} = \frac{s}{4}g.
\]

And so \( Z = \text{Ric} - \frac{s}{4}g \) vanishes iff \( g \) is Einstein.

An oriented manifold is \textit{self-dual} if \( W^- = 0 \).

The simplest examples of compact Einstein manifolds with positive Ricci curvature \((\lambda > 0)\) are provided by the irreducible symmetric spaces of compact type. In dimension 4, this observation yields exactly two orientable examples: \( S^4 = SO(5)/SO(4) \) and \( \mathbb{CP}_2 = SU(3)/U(2) \), both of which actually have positive sectional curvature. The \textit{Fubini-Study metric} is the unique \( U(2) \)-invariant metric on \( \mathbb{CP}_2 = SU(3)/U(2) \) with total volume \( \pi^2/2 \); it is Einstein, and has sectional curvatures \( K(P) \in [1, 4] \). By \textit{homothetically isometric}, we mean isometric after rescaling; in other words, the theorem concludes by asserting the existence of a diffeomorphism \( \Phi : M \to \mathbb{CP}_2 \) such that \( g = \Phi^*c\text{g}_0 \) for some positive constant \( c \).

In this paper, we prove the following.
**Theorem A.** Let $M$ be a smooth compact oriented 4-manifold, and suppose that $g$ is a self-dual Einstein metric on $M$ which has positive sectional curvature. Then $(M, g)$ is homothetically isometric to $S^4$ or to $\mathbb{CP}^2$, equipped with its standard metric.

Let $F : \Lambda^2(T_xM) \to \Lambda^2(T_xM)$ be the Weitzenböck operator given by

$$< F(e_{ij}), e_{kl} > = \text{Ric}(e_i, e_k)\delta_{jk} + \text{Ric}(e_j, e_l)\delta_{ik} - \text{Ric}(e_i, e_l)\delta_{jk} - \text{Ric}(e_j, e_k)\delta_{il} - 2R_{ijkl}.$$ 

This operator satisfies the *magic* Weitzenböck formula, that is,

$$\Delta \omega = -\text{div} \nabla \omega + F(\omega),$$

where $\nabla$ is the covariant differential operator of the Levi-Civita connection of $g$. Moreover, $F$ is a symmetric operator and $\Lambda^+$ and $\Lambda^-$ are $F$-invariant. Then $\ast F = F \ast$, at each point of $M$ we have a decomposition $F = F^+ + F^-$ with respect to the decomposition (1) and a normal form, as in [9] for the curvature tensor $R$.

Let $(M, g)$ be an oriented Einstein 4-manifold. Then, for each $x \in M$, there exists a positively oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_xM$ such that, relative to the corresponding basis $\{e_{12}, e_{34}, e_{13}, e_{42}, e_{14}, e_{23}\}$ of $\Lambda^2_x(M)$.

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ be orthonormal bases of eigenvectors of $F^+$ and $F^-$ respectively, and $f_i^+$ and $f_i^-$, $i = 1, 2, 3$ are the corresponding eigenvalues. It follows easily from the lemma?? that the self-dual -2 forms

$$\alpha_1 = \frac{\sqrt{2}}{2}(e_{12} + e_{34}), \alpha_2 = \frac{\sqrt{2}}{2}(e_{13} - e_{24}), \alpha_3 = \frac{\sqrt{2}}{2}(e_{14} + e_{23})$$

are the eigenvectors of the symmetric operator $F^+$ with corresponding eigenvalues $f_i^+$ and that the anti-self-dual -2 forms

$$\beta_1 = \frac{\sqrt{2}}{2}(e_{12} - e_{34}), \beta_2 = \frac{\sqrt{2}}{2}(e_{13} + e_{24}), \beta_3 = \frac{\sqrt{2}}{2}(e_{14} - e_{24})$$

are the eigenvectors of the symmetric operator $F^-$ with corresponding eigenvalues $f^-$. $K_{ij}$ denote the sectional curvature of the plane $\{e_i, e_j\}$.

Since $M$ is an Einstein 4-manifold, we have $K_{12} = K_{34}, K_{13} = K_{24}, K_{14} = K_{23}$ from [9].
From the definition of $F$, we have

$$f_1^+ = < F(a_1), a_1 >$$

$$= \frac{1}{2} (Ric(e_1) + Ric(e_2) + Ric(e_3) + Ric(e_4) - 2K_{12} - 2K_{34} + 4R_{1234})$$

$$= K_{13} + K_{24} + K_{14} + K_{23} + 2R_{1234}$$

$$= 2(K_{13} + K_{14} + R_{1234}).$$

Similarly, we obtain

$$f_2^+ = 2(K_{12} + K_{14} - R_{1324})$$

$$f_3^+ = 2(K_{12} + K_{13} + R_{1423})$$

$$f_1^- = 2(K_{13} + K_{14} - R_{1234})$$

$$f_2^- = 2(K_{12} + K_{14} + R_{1324})$$

$$f_3^- = 2(K_{12} + K_{13} - R_{1423}).$$

We can therefore state the well-known result as follows.

**Proposition 1.** [8]

The Weitzenböck operator is given in terms of the scalar curvature

$$F^+ = \frac{s^2}{6} - 2W^+$$

$$F^- = \frac{s^2}{6} - 2W^-.$$

**Lemma 1.** Let $(M, g)$ be an oriented Einstein 4-manifold with $W^- = 0$.

Then the norm and determinant of self-dual Weyl curvature tensor satisfy

$$(6) \ |W_+|^2 = \frac{s^2}{6} - 8(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}),$$

$$(8) \text{det} W_+ = \frac{s^3}{6} - 12s(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) + 144K_{12}K_{13}K_{14}.$$  

**Proof.** By the above Proposition, we have
\[ w_1^+ = -\frac{s}{12} + K_{12} + R_{1234} \]
\[ w_2^+ = -\frac{s}{12} + K_{13} - R_{1324} \]
\[ w_3^+ = -\frac{s}{12} + K_{14} + R_{1423} \]
\[ w_1^- = -\frac{s}{12} + K_{12} - R_{1234} \]
\[ w_2^- = -\frac{s}{12} + K_{13} + R_{1324} \]
\[ w_3^- = -\frac{s}{12} + K_{14} - R_{1423} \]

Since \( W^- = 0 \), we obtain
\[ w_1^+ = 2(-\frac{s}{12} + K_{12}) \]
\[ w_2^+ = 2(-\frac{s}{12} + K_{13}) \]
\[ w_3^+ = 2(-\frac{s}{12} + K_{14}). \]

Therefore, we get
\[
|W|^2 = (w_1^+)^2 + (w_2^+)^2 + (w_3^+)^2 \\
= 4(-\frac{s}{12} + K_{12})^2 + 4(-\frac{s}{12} + K_{13})^2 + 4(-\frac{s}{12} + K_{14})^2 \\
= \frac{s^2}{6} - 8(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}).
\]

Also, we obtain
\[
18 \det W^+ = 18w_1^+w_2^+w_3^+ \\
= 144(-\frac{s}{12} + K_{12})(-\frac{s}{12} + K_{13})(-\frac{s}{12} + K_{14}) \\
= \frac{s^3}{6} - 12s(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) + 144K_{12}K_{13}K_{14}.
\]

\[ \square \]
2. The Curvature of 4-Manifolds

The curvatures $W^+$, and $s$ correspond to different irreducible representation of $SO(4)$, so the only invariant quadratic polynomials in the curvature of an oriented self-dual Einstein 4-manifold are linear combinations of $s^2$ and $|W^+|^2$. This observation can be applied, in particular, to simplify the integrands [1, 7, 10] of the 4-dimensional Chern-Gauss-Bonnet

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left[ |W^+|^2 + \frac{s^2}{24} \right] d\mu$$

and Hirzebruch signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W^+|^2 d\mu$$

formulae. Here the curvatures, norms $|\cdot |$, and volume form $d\mu$ are, of course, those of any given Einstein metric $g$ on $M$.

By a simple calculation, we have

$$\chi(M) - \frac{3}{2} \tau(M) = \frac{1}{8\pi^2} \int_M \frac{s^2}{24} d\mu_g > 0.$$  \hspace{1cm} (10)

By our condition, $b_+ > 0$ and $b_- = 0$, so $\chi(M) = 2 + b_+$ and $\tau(M) = b_+ > 0$. Hence

$$\chi(M) - \frac{3}{2} \tau(M) = 2 - \frac{1}{2} b_+ > 0.$$  \hspace{1cm} (10)

We get

$$b_+(M) < 4.$$  \hspace{1cm} (10)

Therefore here are three cases $\tau(M) = 0, \tau(M) = 1, \tau(M) = 2$ and $\tau(M) = 3$, the corresponding Euler characteristic are $\chi(M) = 2, \chi(M) = 3, \chi(M) = 4$ and $\chi(M) = 5$

In case $\tau(M) = 3$, $\chi(M) = 5$, we have

$$3\chi(M) = 5\tau(M).$$

This means that

$$\frac{4}{8\pi^2} \int_M \left[ |W^+|^2 + \frac{s^2}{24} \right] d\mu = \frac{5}{12\pi^2} \int_M |W^+|^2 d\mu.$$
Thus, we obtain
\[ \int_M |W_+|^2 d\mu = \frac{3}{8\pi^2}, \]
which is an obvious contradiction to the inequality [8].

From the result of [5], the case \( \chi(M) = 4, \tau(M) = 2 \) cannot occur. For the case \( \chi(M) = 2, \tau(M) = 0 \), it is well-known that this manifold is isometric to the standard 4-sphere \( S^4 \).

We discuss the remaining case \( \chi(M) = 3, \tau(M) = 1 \). From the above integral relationship between characteristic numbers, we have
\[ \int_M |W_+|^2 d\mu = \frac{1}{24\pi^2}. \]
This is equivalent to the identity
\[ (11) \quad \int_M K_{12}^2 + K_{13}^2 + K_{14}^2 d\mu \int_M 2(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14})d\mu. \]

There are many proofs in this case, but we give a very simple proof. The key observations of §2 were basically point-wise in character. We now turn to some results of a fundamentally global nature.

On the other hand, Derdziński [2, 3, 6] observed the Weitzenböck formula
\[ (12) \quad 0 = \frac{1}{2} \Delta |W^+|^2 + |\nabla W^+|^2 + \frac{s}{2} |W^+|^2 - 18 \det W^+ \]
where \( \Delta \) is again the positive Laplacian and \( \det W^+ \) is the determinant of the bundle endomorphism \( W^+ : \Lambda^+ \to \Lambda^+ \).

Integrating the above Weitzenböck formula and using the equation (11) and (1), we have
\[ 0 = \int_M [ |\nabla W^+|^2 + \frac{s}{2} |W^+|^2 - 18 \det W^+] d\mu \]
We estimate the part of the above integral and use the equation (11) and (1), we have
\[
\int_M -\frac{s^3}{6} + 8s(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) \\
-144K_{12}K_{13}K_{14}d\mu \\
= \int_M \frac{16}{3} [4(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) - K_{12}^2 - K_{13}^2 - K_{14}^2] \\
\times (K_{12} + K_{13} + K_{14}) - 144K_{12}K_{13}K_{14}d\mu \\
= \int_M \frac{32}{3} (K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14})(K_{12} + K_{13} + K_{14}) \\
-144K_{12}K_{13}K_{14}d\mu \\
= \int_M \frac{32}{3} (K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14})(K_{12} + K_{13} + K_{14}) \\
-144K_{12}K_{13}K_{14}d\mu \\
= \int_M \frac{32}{3} (K_{12}^2 + K_{13}^2)K_{14} + (K_{14}^2 + K_{13}^2)K_{12} + (K_{12}^2 + K_{14}^2)K_{13} \\
-112K_{12}K_{13}K_{14}d\mu \\
\geq 0.
\]

Therefore we obtain \( \nabla W^+ \equiv 0 \) and \( |W^+|^2 = \frac{s^2}{23} \). In this case, \((M, g)\) is homothetically isometric to \( \mathbb{CP}_2 \), equipped with its standard Fubini-Study metric.

3. The Proof of Main Theorems

Let \((M, g)\) be a smooth compact oriented self-dual Einstein 4-manifold with positive sectional curvature. By the above discussion, there are two cases. First case is \( \chi(M) = 2, \tau(M) = 0 \). This manifold is isometric to the standard 4-sphere \( S^4 \). The second case is \( \chi(M) = 3, \tau(M) = 1 \). Then \((M, g)\) is homothetically isometric to \( \mathbb{CP}_2 \), equipped with its standard Fubini-Study metric[1, 4].

References


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