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SELF-DUAL EINSTEIN MANIFOLDS OF POSITIVE SECTIONAL CURVATURE

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Abstract. Let (M, g) be a compact oriented self-dual 4-dimensional Einstein manifold with positive sectional curvature. Then we show that, up to rescaling and isometry, (M, g) is S^4 or $\mathbb{CP}_2$, with their cannonical metrics.

1. Introduction and preliminaries

Let (M, g) be an oriented Riemannaian 4-manifold , and let Λ^2 denote the bundles of exterior 2-forms on M. We have an invariant decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{1}$$

as the sum of two vector bundles. Here $\Lambda^\pm$ are the eigenspaces of the Hodge star operator

$$\star : \Lambda^2 \to \Lambda^2,$$

corresponding respectively to the eigenvalue ± 1. Sections of Λ^+ are called *self-dual* 2-forms, whereas sections of Λ^- are called *anti-self-dual* 2-forms. But since the curvature tensor of g may be thought of as a symmetric map $\mathcal{R} : \Lambda^2 \to \Lambda^2$ given by

$$\mathcal{R}(e_{ij}) = \frac{1}{2} \sum_{k,l} R_{ijlk} e_{kl}, \tag{2}$$

where $\{e_i\}$ is a local orthonormal basis of 1-forms , e_{ij} denotes the 2-form $e_i \wedge e_j$ and $R_{ijlk} =< R(e_i, e_j)e_l, e_k >$. Equation (1) gives us a

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decomposition [1, 9] of the curvature into irreducible components

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & Z \\ \hline Z & W^- + \frac{s}{12} \end{array} \right), \tag{3}$$

where the *self-dual* and *anti-self-dual Weyl curvatures* $W^{\pm}$ of Wely curvature $W = W^+ + W^-$ are trace-free as endomorphisms of $\Lambda^{\pm}$. The scalar curvature s is understood here to act by scalar multiplication. On the other hand, Z represents the trace-free Ricci curvature $Ric - \frac{s}{4}g$.

Let $A = W^+ + \frac{s}{12}$ and let $C = W^- + \frac{s}{12}$. They are $trA = trC = \frac{s}{4}$. Then the two components of the Weyl tensor W^+ and W^- are given by $W^+ = A - \frac{s}{12}$ and $W^- = C - \frac{s}{12}$.

Riemannian 4-manifold (M, g) is said to be *Einstein* if it has constant Ricci curvature — i.e. if its Ricci tensor Ric is a constant multiple of the metric:

$$Ric = \frac{s}{4}g. \tag{4}$$

And so $Z = Ric - \frac{s}{4}g$ vanishes iff g is Einstein.

An oriented manifold is *self-dual* if $W^- = 0$.

The simplest examples of compact Einstein manifolds with positive Ricci curvature ($\lambda > 0$) are provided by the irreducible symmetric spaces of compact type. In dimension 4, this observation yields exactly two orientable examples: $S^4 = SO(5)/SO(4)$ and $\mathbb{CP}_2 = SU(3)/U(2)$, both of which actually have positive *sectional* curvature. The *Fubini-Study metric* is the unique $U(2)$-invariant metric on $\mathbb{CP}_2 = SU(3)/U(2)$ with total volume $\pi^2/2$; it is Einstein, and has sectional curvatures $K(P) \in [1, 4]$. By *homothetically isometric*, we mean isometric after rescaling; in other words, the theorem concludes by asserting the existence of a diffeomorphism $\Phi : M \to \mathbb{CP}_2$ such that $g = \Phi^* c g_0$ for some positive constant c.

In this paper, we prove the following.

THEOREM A. *Let M be a smooth compact oriented 4-manifold , and suppose that g is a self-dual Einstein metric on M which has positive sectional curvature. Then (M, g) is homothetically isometric to S^4 or to $\mathbb{CP}_2$, equipped with its standard metric.*

Let $F : \Lambda^2(T_xM) \to \Lambda^2(T_xM)$ be the *Weitzenböck operator* given by

$$\begin{aligned} < F(e_{ij}), e_{kl} > &= Ric(e_i, e_k)\delta_{jk} + Ric(e_j, e_l)\delta_{ik} \\ &- Ric(e_i, e_l)\delta_{jk} - Ric((e_j, e_k)\delta_{il} - 2R_{ijlk}. \end{aligned}$$

This operator satisfies the *magic Weitzenböck formula*, that is,

$$\Delta\omega = -div\nabla\omega + F(\omega), \tag{5}$$

where ∇ is the covariant differential operator of the Levi-Civita connection of g. Moreover, F is a symmetric operator and Λ^+ and Λ^- are F-invariant. Then $\star F = F\star$, at each point of M we have a decomposition $F = F^+ + F^-$ with respect to the decomposition (1) and a normal form ,as in [9] for the curvature tensor R.

Let (M, g) be an oriented Einstein 4-manifold. Then , for each $x \in M$, there exists a positively oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_xM such that, relative to the corresponding basis $\{e_{12}, e_{34}, e_{13}, e_{42}, e_{14}, e_{23}\}$ of $\Lambda^2_x(M)$.

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ be orthonormal bases of eigenvectors of F^+ and F^- respectively , and f_i^+ and f_i^- , $i = 1, 2, 3$ are the corresponding eigenvalues. It follows easily from the lemma**??** that the self-dual -2 forms

$$\alpha_1 = \frac{\sqrt{2}}{2}(e_{12} + e_{34}), \alpha_2 = \frac{\sqrt{2}}{2}(e_{13} - e_{24}), \alpha_3 = \frac{\sqrt{2}}{2}(e_{14} + e_{24})$$

are the eigenvectors of the symmetric operator F^+ with corresponding eigenvalues f_i^+ and that the anti-self-dual -2 forms

$$\beta_1 = \frac{\sqrt{2}}{2}(e_{12} - e_{34}), \beta_2 = \frac{\sqrt{2}}{2}(e_{13} + e_{24}), \beta_3 = \frac{\sqrt{2}}{2}(e_{14} - e_{24})$$

are the eigenvectors of the symmetric operator F^- with corresponding eigenvalues f^- . K_{ij} denote the sectional curvature of the plane $\{e_i, e_j\}$.

Since M is an Einstein 4-manifold, we have $K_{12} = K_{34}, K_{13} = K_{24}, K_{14} = K_{23}$ from [9] .

From the definition of F , we have

$$\begin{aligned} f_1^+ &= < F(\alpha_1), \alpha_1 > \\ &= \frac{1}{2}(Ric(e_1) + Ric(e_2) + Ric(e_3) + Ric(e_4) - 2K_{12} - 2K_{34} + 4R_{1234}) \\ &= K_{13} + K_{24} + K_{14} + K_{23} + 2R_{1234} \\ &= 2(K_{13} + K_{14} + R_{1234}). \end{aligned}$$

Similarly, we obtains

$$\begin{aligned} f_2^+ &= 2(K_{12} + K_{14} - R_{1324}) \\ f_3^+ &= 2(K_{12} + K_{13} + R_{1423}) \\ f_1^- &= 2(K_{13} + K_{14} - R_{1234}) \\ f_2^- &= 2(K_{12} + K_{14} + R_{1324}) \\ f_3^- &= 2(K_{12} + K_{13} - R_{1423}). \end{aligned}$$

We can therefore state the well-known result as follows.

PROPOSITION 1. [8]
The Weitzenböck operator is given in terms of the scalar curvature

$$\begin{aligned} F^+ &= \frac{s}{3} - 2W^+ \\ F^- &= \frac{s}{3} - 2W^-. \end{aligned}$$

LEMMA 1. *Let (M, g) be an oriented Einstein 4-manifold with $W^- =$ 0.*

Then the norm and determinant of self-dual Weyl curvature tensor satisfies

(6) $|W_+|^2 = \frac{s^2}{6} - 8(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}),$

(7)(8) $\det W_+ = \frac{s^3}{6} - 12s(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) + 144K_{12}K_{13}K_{14}.$

Proof. By the above Proposition, we have

$$
\begin{aligned}
w_1^+ &= -\frac{s}{12} + K_{12} + R_{1234} \\
w_2^+ &= -\frac{s}{12} + K_{13} - R_{1324} \\
w_3^+ &= -\frac{s}{12} + K_{14} + R_{1423} \\
w_1^- &= -\frac{s}{12} + K_{12} - R_{1234} \\
w_2^- &= -\frac{s}{12} + K_{13} + R_{1324} \\
w_3^- &= -\frac{s}{12} + K_{14} - R_{1423}.
\end{aligned}
$$

Since $W^- = 0$, we obtain

$$
\begin{aligned}
w_1^+ &= 2(-\frac{s}{12} + K_{12}) \\
w_2^+ &= 2(-\frac{s}{12} + K_{13}) \\
w_3^+ &= 2(-\frac{s}{12} + K_{14}).
\end{aligned}
$$

Therefore ,we get

$$
\begin{gathered}
|W|^2 = (w_1^+)^2 + (w_2^+)^2 + (w_3^+)^2 \\
= 4(-\tfrac{s}{12} + K_{12})^2 + 4(-\tfrac{s}{12} + K_{13})^2 + 4(-\tfrac{s}{12} + K_{14})^2 \\
= \tfrac{s^2}{6} - 8(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}).
\end{gathered}
$$

Also, we obtain

$$
\begin{gathered}
18 \det W_+ = 18 w_1^+ w_2^+ w_3^+ \\
= 144(-\tfrac{s}{12} + K_{12})(-\tfrac{s}{12} + K_{13})(-\tfrac{s}{12} + K_{14}) \\
= \tfrac{s^3}{6} - 12s(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) + 144 K_{12}K_{13}K_{14}.
\end{gathered}
$$

$\square$

2. The Curvature of 4-Manifolds

The curvatures W^+, and s correspond to different irreducible representation of $SO(4)$, so the only invariant quadratic polynomials in the curvature of an oriented self-dual Einstein 4-manifold are linear combinations of s^2 and $|W_+|^2$. This observation can be applied, in particular, to simplify the integrands [1, 7, 10] of the 4-dimensional Chern-Gauss-Bonnet

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left[|W_+|^2 + \frac{s^2}{24}\right] d\mu \tag{8}$$

and Hirzebruch signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 d\mu \tag{9}$$

formulæ. Here the curvatures, norms $|\cdot|$, and volume form $d\mu$ are, of course, those of any given Einstein metric g on M.

By a simple calculation, we have

$$\chi(M) - \frac{3}{2}\tau(M) = \frac{1}{8\pi^2}\int_M \frac{s_g^2}{24} d\mu_g > 0. \tag{10}$$

By our condition , $b_+ > 0$ and $b_- = 0$, so $\chi(M) = 2 + b_+$ and $\tau(M) = b_+ > 0$. Hence

$$\chi(M) - \frac{3}{2}\tau(M) = 2 - \frac{1}{2}b_+ > 0.$$

We get

$$b_+(M) < 4.$$

Therefore here are three cases $\tau(M) = 0, \tau(M) = 1, \tau(M) = 2$ and $\tau(M) = 3$,the corresponding Euler characteristic are $\chi(M) = 2, \chi(M) = 3, \chi(M) = 4$ and $\chi(M) = 5$

In case $\tau(M) = 3, \chi(M) = 5$, we have

$$3\chi(M) = 5\tau(M).$$

This means that

$$\frac{4}{8\pi^2}\int_M \left[|W_+|^2 + \frac{s^2}{24}\right] d\mu = \frac{5}{12\pi^2}\int_M |W_+|^2 d\mu.$$

Thus ,we obtain

$$\int_M |W_+|^2 d\mu = \frac{3}{8\pi^2},$$

which is an obvious contradiction to the inequality[8]

From the result of [5], the case $\chi(M) = 4, \tau(M) = 2$ cannot occur. For the case $\chi(M) = 2, \tau(M) = 0$. it is well-known that this manifold is isometric to the standard 4-sphere S^4.

We discuss the remaining case $\chi(M) = 3, \tau(M) = 1$ .From the above integral relationship between characteristic numbers, we have

$$\int_M |W_+|^2 d\mu = \frac{1}{24\pi^2}.$$

This is equivalent to the identity

$$\int_M K_{12}^2 + K_{13}^2 + K_{14}^2 d\mu \int_M 2(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) d\mu. \tag{11}$$

There are many proofs in this case,but we give a very simple proof.

The key observations of §2 were basically point-wise in character. We now turn to some results of a fundamentally global nature.

On the other hand, Derdziński [2, 3, 6] observed the Weitzenböck formula

$$0 = \frac{1}{2}\Delta|W^+|^2 + |\nabla W^+|^2 + \frac{s}{2}|W^+|^2 - 18 \det W^+ \tag{12}$$

where Δ is again the positive Laplacian and $\det W^+$ is the determinant of the bundle endomorphism $W^+ : \Lambda^+ \to \Lambda^+$.

Integrating the above Weitzenböck formula and using the equation(11) and (1), we have

$$0 \;= \int_M \left[|\nabla W^+|^2 + \tfrac{s}{2}|W^+|^2 - 18 \det W^+\right] d\mu$$

We estimate the part of the above integral and use the equation(11) and (1), we have

$$\begin{aligned}
&\int_M -\tfrac{s^3}{6} + 8s(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) \\
&\qquad -144K_{12}K_{13}K_{14}d\mu \\
&= \int_M \tfrac{16}{3}\left[4(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14}) - K_{12}^2 - K_{13}^2 - K_{14}^2\right] \\
&\qquad \times(K_{12} + K_{13} + K_{14}) - 144K_{12}K_{13}K_{14}d\mu \\
&= \int_M \tfrac{32}{3}(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14})(K_{12} + K_{13} + K_{14}) \\
&\qquad -144K_{12}K_{13}K_{14}d\mu \\
&= \int_M \tfrac{32}{3}(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14})(K_{12} + K_{13} + K_{14}) \\
&\qquad -144K_{12}K_{13}K_{14}d\mu \\
&= \int_M \tfrac{32}{3}(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14})(K_{12} + K_{13} + K_{14}) \\
&\qquad -144K_{12}K_{13}K_{14}d\mu \\
&= \int_M \tfrac{32}{3}(K_{12}^2 + K_{13}^2)K_{14} + (K_{14}^2 + K_{13}^2)K_{12} + (K_{12}^2 + K_{14}^2)K_{13}) \\
&\qquad -112K_{12}K_{13}K_{14}d\mu \\
&\qquad \geq 0.
\end{aligned}$$

Therefore we obtain $\nabla W^+ \equiv 0$ and $|W_+|^2 = \frac{s^2}{24}$. In this case, (M, g) is homothetically isometric to $\mathbb{CP}_2$, equipped with its standard Fubini-Study metric.

3. The Proof of Main Theorems

Let (M, g) be a smooth compact oriented self-dual Einstein 4-manifold with positive sectional curvature. By the above discussion, there are two cases. First case is $\chi(M) = 2, \tau(M) = 0$. This manifold is isometric to the standard 4-sphere S^4. The second case is $\chi(M) = 3, \tau(M) = 1$. Then (M, g) is homothetically isometric to $\mathbb{CP}_2$, equipped with its standard Fubini-Study metric[1, 4].

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