Kangweon-Kyungki Math. Jour. **13** (2005), No. 1, pp. 83–90

DIRECT PROOF OF EKELAND'S PRINCIPLE IN LOCALLY CONVEX HAUSDORFF TOPOLOGICAL VECTOR SPACES

Jong An Park

Abstract. A.H.Hamel proved the Ekeland's principle in a locally convex Hausdorff topological vector spaces by constructing the norm and applying the Ekeland's principle in Banach spaces. In this paper we show that the Ekeland's principle in a locally convex Hausdorff topological vector spaces can be proved directly by applying the famous general principle of H.Brézis and F.E.Browder.

1. Introduction

H.Brézis and F.E.Browder[1] put forward the following general principle in nonlinear functional analysis which unifies the proofs of Ekeland's variational principles[3] and Caristi-Kirk fixed point theorem [2]and Bishop-Phelps lemma and Daneš' drop theorem. Also the invariance theorems for closed sets under flows in metric spaces were proved by the same principle in [1]. We define $S(x) = \{y \in X | y \geq x\}$ in a ordered set X.

Theorem 1.1. [1] *Let X be a Hausdorff topological space with an ordering structure. Let $\psi : X \to \mathbb{R}$ be a function bounded below. Assume*

1. *$S(x)$ is sequentially closed for each $x \in X$;*
2. *$x \leq y$ and $x \neq y$ imply $\psi(y) < \psi(x)$;*
3. *any increasing sequence is relatively compact.*

Then for each $a \in X$ there exists $\bar{a} \in X$ such that $a \leq \bar{a}$ and $\bar{a}$ is maximal.

Received December 22, 2004.
2000 Mathematics Subject Classification: Primary 49J40, 46A03.
Key words and phrases: lower semi-continuous, Minkowski functional, Ekeland's variational principle.

We define the Minkowski functional μ_S of a subset S of the topological vector space X to be

$$\mu_S(x) := \inf\{t > 0 : x \in tS\}.$$

And we define $dom\ \mu_S = \bigcup_{t>0} tS$ and $\mu_S(x) = \infty, x \notin dom\ \mu_S$. Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on a locally convex topological space X. Then we call X be a sequentially complete iff every p_γ-Cauchy sequence converges. Furthermore $f : X \to (\infty, \infty]$ is called proper if $\{x|f(x) < \infty\} \neq \emptyset$ and let $dom\ f = \{x|f(x) < \infty\}$ and it is a sequentially lower semi-continuous function iff for every $c \in \mathbb{R}$, $\{x \in X|f(x) \leq c\}$ is sequentially closed.

LEMMA 1.1. *Let $S \subset X$ be a sequentially closed, bounded and convex set of a Hausdorff locally convex topological space X containing* 0. *Then the followings hold;*

1. $\mu_S : X \to [0, \infty]$ *is an extended-valued proper and sequentially lower semi-continuous function*
2. *for any $x, y \in dom\ \mu_S$ we have $x + y \in dom\ \mu_S$ and*

$$\mu_S(x + y) \leq \mu_S(x) + \mu_S(y)$$

3. *for any $x, y \in dom\ \mu_S, x - y \in dom\ \mu_S$ we have*

$$\mu_S(x) - \mu_S(y) \leq \mu_S(x - y).$$

Proof. 1. Clearly since $0 \in S, \mu_S(0) = 0$, μ_S is proper. We must prove that $C_c = \{x \in X|\ \mu_S(x) \leq c\}$ is sequentially closed for any $c \in [0, \infty]$. Indeed if $c = \infty$, $C_c = X$ is sequentially closed. if $0 \leq c < \infty$, and $x_n \in C_c$ and $x_n \to y$, then $\mu_S(x_n) \leq c$. Hence for each n there exists $\alpha_n, s_n \in S$ such that

$$0 \leq \alpha_n \leq c + \frac{1}{n}, x_n = \alpha_n s_n.$$

Suppose $\alpha_{n_i} \to 0$ for some n_i, then $x_{n_i} = \alpha_{n_i} s_{n_i} \to 0$ because S is bounded. That is, $y = 0$ and $\mu_S(y) = 0 \leq c, y = 0 \in C_c$. Suppose $0 < \delta \leq \alpha_n \leq c + \frac{1}{n}$ for all sufficiently large n, then $\alpha_{n_i} \to \alpha$ for some n_i and $0 < \delta \leq \alpha \leq c$. Therefore

$$\frac{x_{n_i}}{\alpha_{n_i}} = s_{n_i} \to \frac{y}{\alpha}.$$

Since S is sequentially closed, $\frac{y}{\alpha} \in S$ and $y \in \alpha S, \mu_S(y) \leq \alpha \leq c$. Hence $y \in C_c$.

2. Suppose $x, y \in dom\ \mu_S$, then for any $s, t > 0$ such that $\mu_S(x) < s, \mu_S(y) < t$ we have $\frac{x}{s}, \frac{x}{t} \in S$ because S is convex and $0 \in S$. Let $u = x + y$ and from the convexity of S we have
$$\frac{x+y}{u} = (\frac{s}{u})\frac{x}{s} + (\frac{t}{u})\frac{x}{t} \in S.$$
So $x + y \in uS, \mu_S(x + y) \leq u = s + t$. Therefore $\mu_S(x + y) \leq \mu_S(x) + \mu_S(y)$.
3. for any $x, y \in dom\ \mu_S, x - y \in dom\ \mu_S$, by 2
$$\mu_S(x) = \mu_S((x - y) + y) \leq \mu_S(x - y) + \mu_S(y)$$
That is
$$\mu_S(x) - \mu_S(y) \leq \mu_S(x - y).$$

□

We relate the Minkowski functional μ_S with a family of continuous seminorms p_λ.

LEMMA 1.2. *Let $S \subset X$ be a sequentially closed, bounded and convex set of a Hausdorff locally convex topological space X containing* 0. *Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on X . Then there exists $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ a family of positive numbers such that*
$$\frac{1}{\alpha_\lambda} p_\lambda(x) \leq \mu_S(x), x \in dom\ \mu_S.$$

Proof. Since S is bounded in X, for any $\lambda \in \Lambda$ there exists $\alpha_\lambda > 0$ such that $S \subset \alpha_\lambda U_\lambda$, where $U_\lambda := \{x \in X |\ p_\lambda(x) \leq 1\}$. Then $\mu_{U_\lambda} = p_\lambda$. Therefore for any $x \in S$
$$\frac{1}{\alpha_\lambda} p_\lambda(x) = \frac{1}{\alpha_\lambda} \mu_{U_\lambda}(x) \leq \mu_S(x).$$
Suppose $x \in dom\ \mu_S$ and $\mu_S(x) < t$ then $x \in tS, x = ts, t > 0, s \in S$. That is,
$$\frac{1}{\alpha_\lambda} p_\lambda(x) = \frac{1}{\alpha_\lambda} p_\lambda(ts) = \frac{t}{\alpha_\lambda} p_\lambda(s) \leq t\mu_S(s) \leq t$$
because $\mu_S(s) \leq 1$ for any $s \in S$. Hence
$$\frac{1}{\alpha_\lambda} p_\lambda(x) \leq \mu_S(x), x \in dom\ \mu_S.$$

□

2. Main proof

Instead of the methods of proof in A.H.Hamel [4] we prove the following Ekeland's principle in locally convex Hausdorff spaces by using the above general principle directly.

THEOREM 2.1. *Let X be a Hausdorff locally convex topological space that is sequentially complete. Let $f : X \to \mathbb{R}$ be a sequentially lower semi-continuous function, bounded below. Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on X and $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ a family of positive numbers. Then for every $x_0 \in X$ there exists $x^* \in X$ such that*

$$f(x^*) + \gamma_\lambda p_\lambda(x^* - x_0) \leq f(x_0)$$

for all $\lambda \in \Lambda$, and for all $x \in X, x \neq x^*$ there exists $\mu \in \Lambda$ such that

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*).$$

Proof. Let X be equipped with an ordering

$$x \leq y \text{ iff } f(y) - f(x) \leq -\gamma_\mu p_\mu(y - x)\ \forall\mu \in \Lambda.$$

Indeed it is an ordering on X. That is,

- $x \leq x$
- $x \leq y$ and $y \leq x$ imply $x = y$
- $x \leq y$ and $y \leq z$ imply $x \leq z$

Clear $\leq$ is reflexive. If $x \leq y$ and $y \leq x$, then

$$f(y) - f(x) \leq -\gamma_\mu p_\mu(y - x)\ \forall\mu \in \Lambda$$

and

$$f(x) - f(y) \leq -\gamma_\mu p_\mu(x - y)\ \forall\mu \in \Lambda.$$

Therefore we have

$$0 \leq -\gamma_\mu(p_\mu(y - x) + p_\mu(x - y))\ \forall\mu \in \Lambda.$$

Hence $p_\mu(x - y) \leq -p_\mu(y - x)\ \forall\mu \in \Lambda$. Since $p_\mu \geq 0$, we have $p_\mu(x - y) = 0\ \forall\mu \in \Lambda$. Since X is Hausdorff, $x = y$. We showed that $\leq$ is antisymmetric. To prove that $\leq$ is transitive, if $x \leq y$ and $y \leq z$, then

$$f(y) - f(x) \leq -\gamma_\mu p_\mu(y - x)\ \forall\mu \in \Lambda$$

and

$$f(z) - f(y) \leq -\gamma_\mu p_\mu(z - y)\ \forall\mu \in \Lambda.$$

Then

$$f(z) - f(x) \leq -\gamma_\mu(p_\mu(y - x) + p_\mu(z - y))\ \forall\mu \in \Lambda.$$

Since $p_\mu(z-x) \le p_\mu(y-x) + p_\mu(z-y)$,

$$f(z) - f(x) \le -\gamma_\mu(p_\mu(y-x) + p_\mu(z-y)) \le -\gamma_\mu p_\mu(z-x) \ \forall \mu \in \Lambda.$$

Hence $z \le x$. In order to apply the above theorem by replacing $f = \psi$ in [1],

1. Since f is sequentially lower semi-continuous, $S(x)$ is sequentially closed for each $x \in X$;
2. If $x \le y$ and $x \ne y$, then there exists $\mu_0 \in \Lambda$ such that $p_{\mu_0}(y-x) > 0$ because X is Hausdorff. Hence
$$f(y) - f(x) \le -\gamma_{\mu_0} p_{\mu_0}(y-x) < 0, f(y) < f(x);$$
3. for any increasing sequence x_n
$$x_{n+1} \ge x_n, \ f(x_{n+1}) - f(x_n) \le -\gamma_\mu p_\mu(x_{n+1} - x_n) \le 0 \ \forall \mu \in \Lambda.$$
Therefore $\{f(x_n)\}$ is decreasing and bounded below. So $\{f(x_n)\}$ converges. Hence for any $\mu \in \Lambda$, x_n is $p_\mu-$Cauchy. Since X is sequentially complete, $\{x_n\}$ converges in X. Hence $\{x_n\}$ is compact and relatively compact.

From the above principle for any $x_0 \in X$there exists $x^* \in S(x_0)$ such that x^* is maximal. Since $x^* \in S(x_0)$,

$$f(x^*) + \gamma_\lambda p_\lambda(x^* - x_0) \le f(x_0)$$

for all $\lambda \in \Lambda$. Since x^* is maximal, for all $x \ne x^* \in X$ there exists $\mu \in \Lambda$ such that

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*).$$

□

THEOREM 2.2. [4] *Let X be a Hausdorff locally convex topological space that is sequentially complete. Let $f : X \to (\infty, \infty]$ be an extended-valued proper and sequentially lower semi-continuous function, bounded below. Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on X and $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ a family of positive numbers. Then for every $x_0 \in dom f$ there exists $x^* \in X$ such that*

$$f(x^*) + \gamma_\lambda p_\lambda(x^* - x_0) \le f(x_0)$$

for all $\lambda \in \Lambda$, and for all $x \ne x^*$ there exists $\mu \in \Lambda$ such that

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*).$$

Proof. For fixed $x_0 \in dom f$ let us give the same ordering on the following sequentially closed subset C of X

$$C = \{x \in X | f(x) + \gamma_\mu p_\mu(x - x_0) \leq f(x_0)\ \forall \gamma \in \Lambda\}.$$

That is,

$$x \leq y (\in C) \text{ iff } f(y) - f(x) \leq -\gamma_\lambda p_\gamma(y - x)\ \forall \gamma \in \Lambda.$$

Indeed it is an ordering on C that satisfies the three conditions of the above general principle. So C has a maximal x^*. We must prove for all $x \neq x^*$ there exists $\mu \in \Lambda$ such that

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*).$$

If $x \in C(\neq x^*)$, it is not $x^* \leq x$, so the conclusion holds. If $x \notin C(\neq x^*)$ and $x \in dom f$,

$$f(x) + \gamma_\mu p_\mu(x - x_0) > f(x_0)$$

for some $\mu \in \Lambda$. And since $x^* \in C$,

$$f(x^*) + \gamma_\mu p_\mu(x^* - x_0) \leq f(x_0).$$

It follows that

$$f(x^*) + \gamma_\mu p_\mu(x^* - x_0) \leq f(x_0) < f(x) + \gamma_\mu p_\mu(x - x_0).$$

Since $p_\mu(x - x_0) \leq p_\mu(x^* - x_0) + p_\mu(x - x^*)$,

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*)$$

for some $\mu \in \Lambda$. If $x \notin dom f$, clearly the inequality holds. □

THEOREM 2.3. [4] *Let X be a Hausdorff locally convex topological space that is sequentially complete. Let $f : X \to (\infty, \infty]$ be an extended-valued proper and sequentially lower semi-continuous function, bounded below. Let $S \subset X$ be a sequentially closed, bounded and convex set such that $0 \in S$. Then for every $r > 0$, $x_0 \in dom f$ there exists $x^* \in X$ such that*

$$f(x^*) + r\mu_S(x^* - x_0) \leq f(x_0),$$

and for all $x \in X, x \neq x^*$ we have

$$f(x^*) < f(x) + r\mu_S(x - x^*).$$

Proof. Let us fix $x_0 \in dom f, r > 0$. From the sequentially lower semi-continuity of f, μ_S in Lemma 1.1(1), the following subset $X' \subset X$ is sequentially closed.

$$X' = \{x \in X | f(x) + r\mu_S(x - x_0) \leq f(x_0)\}$$

Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on X . Then by Lemma 1.2 there exists $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ a family of positive numbers such that

$$\frac{1}{\alpha_\lambda}p_\lambda(x) \le \mu_S(x), x \in dom\ \mu_S.$$

Let us $\gamma_\lambda = \frac{r}{\alpha_\lambda}$ for all $\lambda \in \Lambda$. We give the following order structure on X'. That is,

$$x \le y(\in X') \text{ iff } f(y) - f(x) \le -\gamma_\lambda p_\lambda(y-x)\ \forall\lambda\in\Lambda.$$

Indeed it is an ordering on C that satisfies the three conditions of the above general principle. So X' has a maximal x^*. We prove that for all $x \in X(\neq x^*)$

$$f(x^*) < f(x) + \mu_S(x - x^*).$$

If $x - x^* \notin dom\ \mu_S$, this inequality holds. Hence we may assume that $x - x^* \in dom\ \mu_S$. Since $x^* \in X'$, $x^* - x_0 \in dom\ \mu_S$. By Lemma 1.1(2) $x - x_0 \in dom\ \mu_S$. If $x \in X'(\neq x^*)$, then $x^* \nleqq x$ and there exists $\mu \in \Lambda$ such that

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*).$$

Since $x - x^* \in dom\ \mu_S$, $\gamma_\mu p_\mu(x - x^*) \le r\mu_S(x - x^*)$ and

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*) \le f(x) + r\mu_S(x - x^*).$$

If $x \notin X'(\neq x^*)$,then

$$f(x) + r\mu_S(x - x_0) > f(x_0)$$

And since $x^* \in X'$,

$$f(x^*) + r\mu_S(x^* - x_0) \le f(x_0).$$

Since $x - x_0 \in dom\ \mu_S$ and $(x - x_0) - (x^* - x_0) = x - x^* \in dom\ \mu_S$, by Lemma 1.1(3) it follows that

$$\mu_S(x - x_0) - \mu_S(x^* - x_0) \le \mu_S(x - x^*).$$

From

$$f(x^*) < f(x) + r(\mu_S(x - x_0) - \mu_S(x^* - x_0)) \le f(x) + \mu_S(x - x^*).$$

□

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Department of Mathematics
Kangwon National University
Chunchon, 200–701, Korea
E-mail: jongan@kangwon.ac.kr