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# DENSITY OF D-SHADOWING DYNAMICAL SYSTEM

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Abstract. In this paper, we give the notion of the $D-$shadowing property, $D-$ inverse shadowing property for dynamical systems. and investigate the density of $D-$shadowing dynamical systems and the $D-$inverse shadowing dynamical systems. Moreover we study some relationships between the $D-$shadowing property and other dynamical properties such as expansivity and topological stability.

## 1. Introduction

In this paper, we introduce the notion of the $D-$shadowing property [resp. $D-$inverse shadowing property] which is a generalization of that of shadowing property [resp. inverse shadowing property] in theory of dynamical systems, and investigate the density of $D-$shadowing dynamical systems and the $D-$inverse shadowing dynamical systems. Moreover we study some relationships between the $D-$shadowing property and other dynamical properties such as expansivity and topological stability.

The shadowing property, which is also well known as the pseudo orbit tracing property, is one of the interesting concepts in the qualitative theory of dynamical systems. The notion of shadowing property of a dynamical system is used to justify the validity of computer simulations of the system, asserting that there is a true orbit of the system close to the computed orbit. Many people have studied the relations between the shadowing property and the classical notions in the qualitative theory of dynamical systems.

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## 2. Preliminaries

In this section, we give the definitions of the basic concepts which are needed in section 3. Let $X$ be a compact metric space with a metric $d$, and let $\mathbb{Z}$ stands for the set of integers and $\mathbb{R}$ be the set of real numbers.

Throughout the paper, we denote $Z(X)$ by the set of all homeomorphisms on $X$ with the $C^0$-metric: for any $f, g \in Z(X)$,

$$d_0(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

DEFINITION 2.1. *A dynamical system(or flow) on $X$ is the triple $(X, \mathbb{R}, F)$, where $F$ is a continuous map from the product space $X \times \mathbb{R}$ into the space $X$ satisfying the following axioms :*

(1) $F(x, 0) = x$ *(identity axiom)*
(2) $F(F(x, s), t) = F(x, s + t)$ *(group axiom)*

*for every $x \in X$ and $s, t \in \mathbb{R}$. We say that the triple $(X, \mathbb{Z}, F)$ which satisfy the above properties* (1) *and* (2) *is a (discrete) dynamical system.*

Remarks 2.2. Let $(X, \mathbb{Z}, F)$ be a (discrete) dynamical system. Then it is easy to show that the map $f : X \to X$ defined by

$$f(x) = F(x, 1) \text{ for all } x \in X$$

is homeomorphism on $X$. Conversely, let $f : X \to X$ be a homeomorphism and define a map

$$F(x, n) = f^n(x) = f \circ \cdots \circ f(x) \quad (n - \text{times})$$

for all $n \in \mathbb{Z}$ and $x \in X$. Then it is not hard to show that $(X, \mathbb{Z}, F)$ is a discrete dynamical system on $X$. Consequently we identify the homeomorphism $f$ with the discrete dynamical system $F$ which it generates.

Let $(X, \mathbb{R}, F)$ be a flow on $X$. For each $t \in \mathbb{R}$, the map $F_t : X \to X$ defined by $F_t(x) = F(x, t)$ is a homeomorphism on $X$ which is call the time $t$-map of $F$.

Definition 2.3. *Let $f \in Z(X)$ and $x \in X$. The set*

$$O(f, x) = \{f^n(x) : n \in \mathbb{Z}\}$$

*is said to be the orbit of $f$ through $x \in X$.*

Definition 2.4. *Let $\delta > 0$ be an arbitrary number. A $\delta$-pseudo orbit of $f$ is a sequence of points $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ such that*

$$d(f(x_n), x_{n+1}) < \delta$$

*for all $n \in \mathbb{Z}$.*

The notion of a pseudo orbit plays an important role in the general qualitative theory of dynamical systems. Usually, a $\delta$-pseudo orbit is a natural model of computer output in a process of numerical investigation of the dynamical system $f$ in $X$. In this case, the value $\delta$ measures one step errors of the method and round-off errors. It is also used to define some types of invariant sets such as the chain recurrent set or chain prolongation sets(see [5], [6], [11]).

Definition 2.5. *We say that a pseudo orbit $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is $\delta$-shadowed by a point $x \in X$ if the inequality*

$$d(f^n(x), x_n)) < \delta$$

*for all $n \in \mathbb{Z}$, holds.*

Thus the existence of a shadowing point for a pseudo orbit $\xi$ means that $\xi$ is close to a real orbit of $f$.

Definition 2.6. *A homeomorphism $f \in Z(X)$ is said to have the shadowing property (or the pseudo orbit tracing property) if for every $\epsilon > 0$ there exists $\delta > 0$ such that any $\delta$-pseudo orbit $\{x_n\}_{n \in \mathbb{Z}}$ in $X$ is epsilon-shadowed by some point $x \in X$ : i.e.,*

$$d(f^n(x), x_n) \leq \epsilon$$

*for all $n \in \mathbb{Z}$.*

Definition 2.7. *A dynamical system $f \in Z(X)$ is said to have the $H$-shadowing property if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ for every $g \in Z(X)$, then any $g$-orbit is $\epsilon$-shadowed by a $f$-orbit : for every $x \in X$, there exists $x_0 \in X$ such that*

$$d(f^n(x_0), g^n(x)) \leq \epsilon$$

*for all $n \in \mathbb{Z}$.*

The theory of shadowing was developed intensively in recent years and became a significant part of the qualitative theory of dynamical systems containing a lot of interesting and deep results (see [8]).

Definition 2.8. *A homeomorphism $f \in Z(X)$ is said to be expansive if there is a constant $e > 0$ such that if*

$$d(f^n(x), f^n(y)) \leq e$$

*for all $n \in \mathbb{Z}$, then $x = y$. Such a number $e$ is called an expansive constant of $f$.*

Definition 2.9. *A homeomorphism $f \in Z(X)$ is said to be topologically stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$, $g \in Z(X)$, then there is a continuous surjection $h : X \to X$ with $f \circ h = h \circ f$ and $d_0(h, I_X) < \epsilon$, where $I_X : X \to X$ stands for the identity homeomorphism. The map $h$ is called a semiconjugacy from $f$ to $g$ .*

Remarks 2.10. It is well known that if $M$ is a compact smooth manifold and $f \in Z(M)$ is topologically stable then it has the shadowing property. Moreover, it was proved that if $f \in Z(M))$ is an expansive homeomorphism which has the shadowing property then it is topologically stable.

However, the above results do not hold in general if $M$ is not compact smooth manifold.

## 3. Density of $D-$Shadowing Dynamical Systems

In this section, we introduce the notion of $D-$shadowing property and $D-$inverse shadowing property for dynamical systems and study some relationships between the $D-$shadowing property and other dynamical properties such as shadowing property, expansivity and topological stability.

Recently Diamond et al ([2]) obtained a necessary and sufficient condition under which a homeomorphism on a compact smooth manifold has the shadowing property. Let $M$ be a compact smooth manifold. A homeomorphism $f$ on $M$ has the shadowing property if and only if $f$ has the $H$-shadowing property. This theorem can be used to motivate the notion of another shadowing property for dynamical systems on metric spaces as follows.

Definition 3.1. *Let $D$ be a subset of $Z(X)$. Then a dynamical system $f \in Z(X)$ is said to have the $D-$shadowing property if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ for $g \in D$, then any $g$-orbit is $\epsilon$-shadowed by a $f$-orbit : i.e., for every $x \in X$, there exists $x_0 \in X$ such that*

$$d_(f^n(x_0), g^n(x)) \leq \epsilon$$

*for all $n \in \mathbb{Z}$.*

Definition 3.2. *Let $D$ be a subset of $Z(X)$. Then a dynamical system $f \in Z(X)$ is said to have the $D-$inverse shadowing property if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ for $g \in D$, then any $f$-orbit is $\epsilon$-shadowed by a $g$-orbit : i.e., for every $x \in X$, there exists $x_0 \in X$ such that*

$$d(f^n(x), g^n(x_0)) \leq \epsilon$$

*for all $n \in \mathbb{Z}$.*

Remarks 3.3. If $f \in Z(X)$ is a homeomorphism with $d_0(f, g) < \delta$, that is $g$ is a small perturbation of $f$, then we can see that every $g$-orbit $O(g, x) = \{g^n(x) : n \in \mathbb{Z}\}$ is $\delta$-pseudo orbit of $f$. In fact we have

$$d(f(g^n(x)), g^{n+1}(x)) \leq \delta$$

for all in $n \in \mathbb{Z}$.

If $Z(X) = D$, then the $D-$shadowing property is equal the $H$-shadowing property. If there is no $g \in D$ with $d_0(f, g) < \delta$, then we say that $f$ has the $D-$shadowing property.

Let $M$ be a compact manifold with a metric $d$, and let $M^*$ be the space of nonempty closed subsets of $M$ with the Hausdorff metric $\bar{d}$: i.e., for any $A, B \in M^*$,

$$\bar{d}(A, B) = \max\{\sup\{d(a, B) : a \in A\},\ \sup\{d(A, b) : b \in B\}\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. Then $M^*$ become a compact metric space. Let $M^{**}$ be the set of all nonempty closed subsets of $M^*$ with the Hausdorff metric $d_H$ .

Fix a system $f \in Z(M)$ and a point $x \in M^0$. The set $\overline{O(f, x)}$ is an element of $M$. We define the map

$$\Theta : Z(M) \to M^{**}$$

by $\theta(f)\overline{\{\overline{O(f, x)} : x \in M\}}$.

A dynamical system $f \in Z(M)$ is tolerance stable if the map

$$\Theta : Z(M) \to M^{**}$$

is continuous at $f$ (see [10]). A map $h : M \to M^*$ is said to be upper (or lower) semi-continuous at $x \in M$ if for any $\epsilon > 0$ there exists a neighborhood $U$ of $x$ such that for any $z \in U$ we have

$$h(z) \subset B_\epsilon(h(x)) \text{ ( or } h(x) \subset B_\epsilon(h(z)))$$

respectively, where $B_\epsilon(A) = \{z \in M : d(x, z) < \epsilon \text{ for some } x \in A\}$. A map $h : M \to M^*$ is continuous at $x \in M$ if and only if $h$ is upper and lower semicontinuous at $x$ (see [7]). Then we can get the following theorem.

THEOREM 3.4. *The restriction $\Theta_{|D} : D \to M^{**}$ is continuous at $f$ if $f$ has the $D-$shadowing property and $D-$inverse shadowing property.*

*Proof.* Let $D$ be a subset of $Z(M)$, and it $\epsilon > 0$ be arbitrary. Since $f$ has the $D-$shadowing property and $D-$inverse shadowing property, we can choose $\delta > 0$ such that if $d_0(f, g) < \delta$ and $x \in M$, then

$$d(f^n(x_0), g^n(x)) \leq \frac{\epsilon}{3} \text{ and } d(f^n(x), g^n(y_0)) \leq \frac{\epsilon}{3}$$

for some $x_0, y_0 \in M$ and all $n \in \mathbb{Z}$. The proof is completed by showing that $\Theta(g) \subset B_\epsilon(\Theta(f))$ and $\Theta(f) \subset B_\epsilon(\Theta(g))$, where $B_\epsilon(\Theta(f)) = \{A \in M^* : \bar{d}(A, B) < \epsilon$ for some $B \in \Theta(f)\}$.

First we show that $\Theta(g) \subset B_\epsilon(\Theta(f))$. Let $A \in \Theta(g)$. Then there exists $x \in M$ such that $\bar{d}(A, \overline{O(g,x)}) < \frac{\epsilon}{2}$. For the point $x \in M$, we select $x_0 \in M$ satisfying

$$d(f^n(x_0), g^n(x)) \leq \frac{\epsilon}{3}$$

for all $n \in \mathbb{Z}$. then we have $\bar{d}(\overline{O(f, x_0)}, \overline{O(g,x)}) < \frac{\epsilon}{2}$, and so

$$\bar{d}(A, \overline{O(f, x_0)}) < \epsilon.$$

This means that $\Theta(g) \subset B_\epsilon(\Theta(f))$, and hence the map $\Theta$ is upper semicontinuous at $f$.

Next we show that $\Theta(f) \subset B_\epsilon(\Theta(g))$. Let $A \in \Theta(f)$. Then there exists $x \in M$ such that $\bar{d}(A, O(\bar{f}, x)) < \frac{\epsilon}{2}$. For the point $x \in M$, we select $y_0 \in M$ satisfying

$$d(f^n(x), g^n(y_0)) \leq \frac{\epsilon}{3}$$

for all $n \in \mathbb{Z}$. then we have $\bar{d}(O(\bar{f}, x), O(\bar{g}, y_0)) < \frac{\epsilon}{2}$, and so

$$\bar{d}(A, O(\bar{g}, y_0)) < \epsilon.$$

This means that the map $\Theta$ is lower semicontinuous at $f$. Hence the map $\Theta$ is continuous at $f$. □

Tolerance Stability Conjecture : Let $M$ be a compact manifold. For any $\Gamma$-set $D$ of $Z(M)$, there exists a residual set $D_0 \subset D$ such that every $f \in D_0$ is tolerance $D-$stable : i.e.

$$\Theta_{|D} : D \to M^{**}$$

is continuous at every point of $D_0$.

Definition 3.5. *A subset $D$ of $Z(M)$ is called a $\Gamma$-set if $D$ has a topology which is finer than the subspace topology on $D \subset Z(M)$.*

Example 3.6. *The set of $C^r$-diffeomorphisms on $M$, $Diff^r(M)$ with the $C^r$ topology ($r \geq 1$) is a $\Gamma$-set.*

Question 3.7. *For any $\Gamma$-set $D$ of $Z(M)$, is there a residual set $D_0 \subset D$ such that every $f \in D_0$ is $D$-shadowing property or $D$-inverse shadowing property ?*

As a partial answer, we have the following theorem.

Theorem 3.8. *Let $M$ be a compact two dimensional space. Then there exists $\Gamma$-sets $D$, $D_1$ and $D_2$ in $Z(M)$ such that :*

(1) *$D_1$ and $D_2$ are dense in $D$, and $D$ is the disjoint union of $D_1$ and $D_2$ ; i.e. $D_1 \cap D_1 = \emptyset$, $D_1 \cup D_1 = D$.*
(2) *Every element $f \in D_1$ does not have the $D$-shadowing property.*
(3) *Every element $f \in D_2$ does not have the $D$-inverse shadowing property.*

*Proof.* Let $M$ be a two dimensional space, and identify a coordinate neighborhood of $M$ with $\mathbb{R}^2$. The set of flows we construct will have support in $\mathbb{R}^2$, so we forget the rest of $M$.

Set

$$A_0 = \{(x, y) : 0 \leq x \leq 5,\ 0 \leq y \leq 1\},$$
$$A_1 = \{(x, y) : 2 \leq x \leq 3,\ 0 \leq y \leq 1\}, \text{ and}$$
$$A_2 = (\mathbb{R}^2 - \mathrm{Int}A_0) \cup \{(i, \frac{1}{n}) : i = 1 \text{ or } 4,\ n = 1, 2, \}.$$

Let $E : \mathbb{R}^2 \to T\mathbb{R}^2$ be the unit constant vector field, and let $d$ be the usual metric on $\mathbb{R}^2$ . Define $\mathcal{F}$ to be the set of flows generated by vector fields $X$ on $\mathbb{R}^2$ which satisfy the following conditions:

(1) $X_p = d(p, A_2)$ for $p \notin IntA_1$
(2) there is a homeomorphism $h : A_1 \to A_1$ such that $h$ is the identity map on $[2, 3] \times \{0, 1\}$, $h([2, 3] \times r)$ is an integral curve of $X_{|A_1}$ for each $r \in (0, 1)$. See the following figure :

We consider $\mathcal{F}$ with the standard $C^0$-topology for flows : i.e. for any $\phi, \psi \in \mathcal{F}$ with the metric $\rho_0$,

$$\rho_0(\phi, \psi) = \sup\{(\phi(x,t), \psi(x,t)) : x \in M, -1 \leq t \leq 1\}.$$

Let $\mathcal{F}_1$ be the set of $\phi \in \mathcal{F}$, where $\phi$ has no orbits connecting the fixed points $(1, \frac{1}{n})$ and $(4, \frac{1}{m})$, $n, m > 0$ and let

$$\begin{aligned} D &= \{\phi_1 \in Z(M) : \phi_1 \text{ is the time 1-map of } \phi \in \mathcal{F}\}, \\ D_1 &= \{\phi_1 \in Z(M) : \phi_1 \text{ is the time 1-map, } \phi \in \mathcal{F}\}, \text{ and} \\ D_2 &= D - D_1. \end{aligned}$$

Then the sets $D$, $D_1$ and $D_2$ are required -set in $Z(M)$. First we show that $D_1$ and $D_2$ are dense in $D$, and $D$ is the disjoint union of $D_1$ and $D_2$ : In fact, with a small perturbation, we can break all connection between points $(1, \frac{1}{n})$ and $(4, \frac{1}{m})$. This implies that $D_1$ is dense in $D$. Moreover, with a small perturbation, we can make some connection for large $n, m$. This implies that $D_2$ is dense in $D$. By definition, it is clear that

$$D_1 \cap D_2 = \emptyset, \quad D_1 \cup D_2 = D.$$

Next we claim that every element $f \in D_1$ does not have the $D$-shadowing property : Let $\epsilon = \frac{1}{2}$. For every $\delta > 0$, there exists $g \in D_2$

with $d_0(f,g) < \delta$ and $x \in M$ such that for every $x_0 \in M$,

$$d(g^n(x), f^n(x_0)) \geq \epsilon$$

for some $n \in \mathbb{Z}$. i.e. for any $\delta-$trajectory $\xi = \{g^n(x) : n \in \mathbb{Z}\}$ of $f$ is not $\epsilon-$shadowed by $f-$orbit and any point of $M$. This means that $f$ does not have the $D-$shadowing property.

Finally we claim that every element $f \in D_2$ does not have the $D-$inverse shadowing property : Let $\epsilon = \frac{1}{2}$. For any $\delta > 0$, there exists $g \in D_1$ with $d_0(f,g) < \delta$ and $x \in M$ such that for every $x_0 \in M$,

$$d(g^n(x_0), f^n(x)) \geq \epsilon$$

for some $n \in \mathbb{Z}$. i.e. for any $\delta-$trajectory $\xi = \{f^n(x) : n \in \mathbb{Z}\}$ of $g$ is not $\epsilon$-shadowed by any point of $M$. This means that $f$ does not have the $D-$inverse shadowing property. Consequently we complete the proof of the theorem. □

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