

NUMERICAL METHODS FOR A STIFF PROBLEM ARISING FROM POPULATION DYNAMICS

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ABSTRACT. We consider a model of population dynamics whose mortality function is unbounded. We note that the regularity of the solution depends on the growth rate of the mortality near the maximum age. We propose Gauss-Legendre methods along the characteristics to approximate the solution when the solution is smooth enough. It is proven that the scheme is convergent at fourth-order rate in the maximum norm. We also propose discontinuous Galerkin finite element methods to approximate the solution which is not smooth enough. The stability of the method is discussed. Several numerical examples are presented.

1. Introduction

We consider the following nonlinear Gurtin-MacCamy system:

$$\begin{aligned}
 (1.1) \quad & \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu(a, p(t))u = 0, \quad 0 < a < a_+, \quad t > 0, \\
 & u(0, t) = \int_0^{a_+} \beta(a, p(t))u(a, t)da, \quad t > 0, \\
 & u(a, 0) = u_0(a), \quad 0 \leq a < a_+, \\
 & p(t) = \int_0^{a_+} u(a, t)da, \quad t \geq 0.
 \end{aligned}$$

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Here, the function $u(a, t)$ denotes the age-specific density of individuals of age a at time t and $p(t)$ denotes the total population at time t . The demographic parameters $\mu = \mu(a, p) \geq 0$ and $\beta = \beta(a, p) \geq 0$ are the death and the birth rates, respectively, at age a with the total population p . System (1.1) describes the evolution of the age density $u(a, t)$ of a population with a maximum age $a_+ < \infty$, whose growth is regulated by the vital rates $\beta(a, p)$ and $\mu(a, p)$ (see for example [2, 3, 9]).

In recent years, several numerical methods have been proposed for the approximation of the solution of (1.1) (see for example [1] and the references cited therein). There, the maximum age a_+ was assumed to be infinite and/or the mortality function μ was assumed to be Lipschitz continuous.

In this paper we consider the case that all individuals of the population have a finite life-span and thus the maximum age a_+ is finite. It is known that, when the maximum age is infinite, the solution is smooth and its derivatives are bounded if certain compatibility conditions at origin are satisfied ([2]). However, in the case that the maximum age is finite, the solution may happen to be stiff depending on the mortality μ , even if those compatibility conditions are satisfied.

We note that the regularity of the solution depends on the growth rate of μ near the maximum age. We propose two schemes for the approximation of the solution to (1.1); one for the smooth solution and the other one for the nonsmooth solution. For the smooth solution, we propose Gauss-Legendre methods (fourth order implicit Runge-Kutta methods of two stage) to the integration of the ODE along the characteristics, whose collocation points are zeros of the linearly transformed Legendre monic polynomial. Stability of the method is then proved using the nonnegative property of numerical solutions. In the last two sections, we discuss discontinuous Galerkin finite element methods for the approximation of the non-smooth solution.

The organization of the remainder of this article is as follows. In the next section, we describe the hypotheses and investigate regularity of the solution to problem (1.1), which is needed in Section 3 for the proof of an optimal convergent rate. In Section 3, we present a collocation method. We prove that the numerical solutions are non-negative. We also show that the scheme is stable and it has an optimal

convergence rate of fourth order. In Section 4, we report results from numerical experiments with some remarks. In Section 5, we introduce a discontinuous Galerkin finite element methods for the approximation of the non-smooth solution. In the last section, we provide an L^2 error analysis and the stability of the method.

2. Hypotheses and regularity of solutions

Throughout the paper, we assume that the death process takes the separable form:

$$\mu(a, p) = m(a) + M(a, p),$$

where the function m is called the natural mortality and the function M is called the external mortality caused by external “force”. (See [3] and references cited therein for biological meanings.)

We shall also assume the following hypotheses: For some integer $k \geq 0$,

- (H1) the nonnegative function u_0 belongs to $C^k([0, a_+))$ and has a compact support in $[0, a_+)$,
- (H2) $\beta \in C^{k+1}([0, a_+) \times [0, \infty))$, $\frac{\partial \beta}{\partial p} \in C^k([0, a_+) \times [0, \infty))$ are bounded for bounded p , and $0 \leq \beta(a, p) \leq \bar{\beta}$ for some positive constant $\bar{\beta}$, and $\beta(\cdot, t)$ has a compact support in $[0, a_+)$,
- (H3) $0 \leq \mu(a, p) = m(a) + M(a, p)$, where $\int_0^{a_+} m(a) da = +\infty$, $M \leq \bar{M}$ for some positive constant $\bar{M}$, $m \in C^{k+1}([0, a_+))$, and $M \in C^{k+1}([0, a_+) \times [0, \infty))$, $\frac{\partial M}{\partial p} \in C^k([0, a_+) \times [0, \infty))$ are bounded for bounded p .

Under the assumptions (H1)–(H3), it is known ([3]) that problem (1.1) has a unique nonnegative solution, global in time.

Concerning the natural mortality m , we also assume the following growth rates near a_+ [5], which will be used to prove the regularity of solutions of (1.1):

- (H4) For a near a_+ , m takes the form $m(a) = c/(a_+ - a)$, for some $c \geq k + 1$, or $m(a) = c/(a_+ - a)^\alpha$, for some $\alpha > 1$, $c > 0$,
- (H5) there exist positive constants $a_* < a_+$ and $\bar{m}$ such that $0 \leq m(a) \leq \bar{m}$ for $a \in [0, a_*]$ and m is monotone increasing in $[a_*, a_+)$ with $m(a_*) = \sup_{a \in [0, a_*]} \{m(a)\}$.

We now investigate regularity of the solution of (1.1), which is needed in Section 3 for the proof of optimal rate of convergence.

THEOREM 2.1. *Assume that (H1)-(H5) hold. Then, for each $t \geq 0$, $u(a, t)$ approaches zero as a tends to a_+ and $\frac{\partial^l u}{\partial a^l}$ remains bounded, for $0 \leq l \leq k+1$. Furthermore, for each $t > 0$, $\frac{\partial^l u}{\partial a^l}$ and $\frac{d^l m}{da^l} u$, $0 \leq l \leq k+1$, approach zero as a tends to a_+ . If m takes the form $m = \frac{c}{a_+ - a}$ near a_+ , then the last statement holds only when $c > k + 1$.*

From Theorem 2.1, we have the following regularity result for the solution $u(a, t)$ of system (1.1).

THEOREM 2.2. *Assume that (H1)-(H5) hold. Then the solution u of (1.1) belongs to $C^{k+1}([0, a_+] \times [0, T] \setminus \{(a, t) \mid a = t\})$ for some $T > 0$. If further the compatibility conditions given below are satisfied, then the solution u of (1.1) belongs to $C^1([0, a_+] \times [0, T])$.*

$$\begin{aligned} u_0(0) &= \int_0^{a_+} \beta(a, p_0) u_0(a) da, \quad p_0 = \int_0^{a_+} u_0(a) da, \\ u'_0(0) &= -\{\mu(0, p_0) + \beta(0, p_0)\} u_0(0) \\ &\quad - \int_0^{a_+} \left\{ \frac{\partial \beta}{\partial a}(a, p_0) + \frac{\partial \beta}{\partial p}(a, p_0) p^* - \beta(a, p_0) \mu(a, p_0) \right\} u_0(a) da, \end{aligned}$$

where $p^* = u_0(0) - \int_0^{a_+} \mu(a, p_0) u_0(a) da$.

3. The Numerical Method and Convergence

We consider the cohort function $v_\tau(t) = u(t + \tau, t)$, $t \geq t_\tau$, where $t_\tau = \max\{0, -\tau\}$, corresponding to age τ and it keeps track along the characteristic of individuals of the population who are initially of age τ as time evolves. Then, noting that

$$\frac{d}{dt} v_\tau(t) = -\mu(t + \tau, p(t)) v_\tau(t), \quad t \geq t_\tau,$$

we have the following:

$$(3.1) \quad \frac{d}{dt} u(t + \tau, t) = -\mu(t + \tau, p(t)) u(t + \tau, t), \quad t \geq t_\tau.$$

Let us denote by $h = \Delta t$ the age-time discretization parameter chosen so that $J = a_{\dagger}/h$ is a positive even integer. Let $T > 0$ be the final time and let $N = [T/h]$ be a positive integer. Let us ease the notation by setting $a_j = jh, 0 \leq j \leq J, t^n = nh, 0 \leq n \leq N$, and $f_j^n = f(a_j, t^n)$ for a function $f(a, t)$ of age and/or time. For the approximations U_j^n of $u(a_j, t^n)$, we first note, from the solution of the system (1.1), that $u(a_{\dagger}, t) = 0$. We thus let $U_J^n \equiv 0$.

For the approximation of the differential equation of problem (1.1), we apply a modified collocation method (or modified fourth order implicit Runge-Kutta method of two stage) along the characteristics. That is, for $3 \leq n \leq N - 1, 0 \leq j \leq J - 2$, we use the following algorithm:

$$\begin{aligned}
 (3.2) \quad & \xi_1 = U_j^n - h\{1/4 \tilde{\mu}_{j+c_1}^{n+c_1} \xi_1 + (1/4 - \sqrt{3}/6) \tilde{\mu}_{j+c_2}^{n+c_2} \xi_2\}, \\
 & \xi_2 = U_j^n - h\{(1/4 + \sqrt{3}/6) \tilde{\mu}_{j+c_1}^{n+c_1} \xi_1 + 1/4 \tilde{\mu}_{j+c_2}^{n+c_2} \xi_2\}, \\
 & U_{j+1}^{n+1} = U_j^n - h/2\{\tilde{\mu}_{j+c_1}^{n+c_1} \xi_1 + \tilde{\mu}_{j+c_2}^{n+c_2} \xi_2\},
 \end{aligned}$$

where $\tilde{\mu}_{j+c_i}^{n+c_i} = \mu(a_j + c_i h, P^{n+c_i}), c_1 = 1/2 - \sqrt{3}/6, c_2 = 1/2 + \sqrt{3}/6$,

$$\begin{aligned}
 (3.3) \quad & P^{n+c_1} = \{55/24 - (107/216)\sqrt{3}\}P^n - \{59/24 - (71/72)\sqrt{3}\}P^{n-1} \\
 & + \{37/24 - (47/72)\sqrt{3}\}P^{n-2} - \{3/8 + (35/216)\sqrt{3}\}P^{n-3},
 \end{aligned}$$

$$\begin{aligned}
 (3.4) \quad & P^{n+c_2} = \{55/24 + (107/216)\sqrt{3}\}P^n - \{59/24 + (71/72)\sqrt{3}\}P^{n-1} \\
 & + \{37/24 + (47/72)\sqrt{3}\}P^{n-2} - \{3/8 - (35/216)\sqrt{3}\}P^{n-3}.
 \end{aligned}$$

And for the approximation of the integral equations of problem (1.1), we use the Simpson's rules. That is, for $0 \leq n \leq N - 1$, we use the

followings:

(3.5)

$$\begin{aligned}
 U_0^{n+1} &= \frac{h}{3 - \tilde{\beta}_0^{n+1}h} [4\tilde{\beta}_1^{n+1}U_1^{n+1} + \tilde{\beta}_2^{n+1}U_2^{n+1} \\
 &\quad + \sum_{k=1}^{\frac{J}{2}-1} (\tilde{\beta}_{2k}^{n+1}U_{2k}^{n+1} + 4\tilde{\beta}_{2k+1}^{n+1}U_{2k+1}^{n+1} + \tilde{\beta}_{2k+2}^{n+1}U_{2k+2}^{n+1})], \\
 P^{n+1} &= \frac{h}{3} \{U_0^{n+1} + 4U_1^{n+1} + U_2^{n+1} + \sum_{k=1}^{\frac{J}{2}-1} (U_{2k}^{n+1} + 4U_{2k+1}^{n+1} + U_{2k+2}^{n+1})\},
 \end{aligned}$$

where $\tilde{\beta}_j^n = \beta(a_j, P^n)$. For the initialization, we may use the schemes based on the method of characteristics ([4], for example) together with Richardson extrapolations. We now prove that the approximate solutions U_j^n , P^n of the problem are nonnegative.

THEOREM 3.1. *Assume that (H1)-(H5) hold. If $0 < h < h^*$ and $U_j^n \geq 0$, $P^n \geq 0$ for $0 \leq n \leq 3$, $j \geq 0$, then $U_j^n \geq 0$, $P^n \geq 0$ for all $0 \leq n \leq N$ and $0 \leq j \leq J$, where $h^* = \min \left\{ \frac{2}{M+m}, \frac{1}{6M} \right\}$ for $\alpha = 1$ and $h^* = \min \left\{ \frac{2}{M+m}, \frac{1}{6M}, [c\{\sqrt{3}((\frac{1}{3})^{1/\alpha} - (\frac{3}{11})^{1/\alpha})\}^\alpha]^{1/(\alpha-1)} \right\}$ for $\alpha > 1$.*

Next, we prove the stability. In view of Theorem 3.1, we will prove that the numerical solutions are bounded above by a constant.

Throughout the paper C will denote a generic positive constant which is independent of h and not necessarily the same at each occurrence.

THEOREM 3.2. *Assume that (H1)-(H5) hold. If $0 < h \leq h^{**} = \min\{h^*, \frac{1}{\beta}\}$ and $U_j^n \geq 0$, $P^n \geq 0$ for $0 \leq n \leq 3$, $j \geq 0$, then there exists a positive constant $C = C(h^{**}, T)$ such that $0 \leq U_j^n \leq C$, $0 \leq P^n \leq C$ for all $0 \leq n \leq N$ and $0 \leq j \leq J$.*

We now show that the implicit Runge-Kutta formulation (3.2)–(3.5) can be viewed as a discretized collocation approximation. As far as actual computation is concerned, implicit Runge-Kutta formulation is preferable. The advantage of collocation formulation is that it lends itself conveniently to analysis.

We note that c_1 and c_2 of (3.2)–(3.4) are zeros of the linearly transformed Legendre (monic) polynomial $P_2(t) = t^2 - t + 1/6$ that is the

orthogonal polynomial for the weight function $\omega(t) \equiv 1$, $0 \leq t \leq 1$. We also note that we may write $U_j^n = U_{n+k}^n$, for $0 \leq j \leq J$, $-n \leq k \leq J-n$. We now consider the following collocation approximation along the characteristics: Let $V_k^0 = u_0(kh)$ and $V_0^n = u_0^n$, for $0 \leq k \leq J$, $0 \leq n \leq N$. Then, for $0 \leq n \leq N-1$, $-n \leq k \leq J-n-2$, we find a second degree polynomial along the characteristics, $v(t+kh, t)$ for $t^n \leq t \leq t^{n+1}$ satisfying

$$(3.6) \quad \begin{aligned} v_{n+k}^n &= V_{n+k}^n, \\ \frac{d}{dt}v(t^n + kh + c_l h, t^n + c_l h) &= \\ &- \mu(t^n + kh + c_l h, p(t^n + c_l h))v(t^n + kh + c_l h, t^n + c_l h), \quad l = 1, 2, \end{aligned}$$

and we set $V_{n+k+1}^{n+1} = v_{n+k+1}^{n+1}$. The implicit Runge-Kutta formulation (3.2)–(3.5) is identical to the collocation method (3.6). We can now prove that the approximate solution by the algorithm (3.2)–(3.5) converges to the true solution at a fourth order rate. Let $E_j^n = u_j^n - U_j^n$, $e_j^n = u_j^n - V_j^n$, $\tilde{e}_j^n = V_j^n - U_j^n$, for $0 \leq n \leq N$, $0 \leq j \leq J$, and $\rho^n = p^n - P^n$, $0 \leq n \leq N$, and let

$$\|\chi^n\|_{\ell^1} = \sum_{j=0}^J |\chi_j^n| h, \quad n \geq 0.$$

THEOREM 3.3. *Assume that (H1)–(H5) hold with $k = 3$. Let $u \in C^4([0, a_+] \times [0, T])$ and let u have bounded derivatives through fifth order in the characteristic direction $\tau = \frac{1}{\sqrt{2}}(1, 1)$. If $T = Nh$ is fixed, $0 < h \leq h^{**}$, and if we assume that $|E_j^n| = \mathcal{O}(h^4)$, $|\rho^n| = \mathcal{O}(h^4)$, for $0 \leq n \leq 3$, $j \geq 0$, then there exists a positive constant C , independent of h , such that*

$$\max_{0 \leq n \leq N} \max_{0 \leq j \leq J} |u_j^n - U_j^n| \leq Ch^4 \quad \text{and} \quad \max_{0 \leq n \leq N} |p^n - P^n| \leq Ch^4.$$

4. Numerical Results

In this section we present several numerical results. In all tests we computed the order of convergence of the algorithms by the well-known

formula:

$$r(h) = \frac{\log \frac{E(h)}{E(h/2)}}{\log 2},$$

where $E(h)$ is the approximation error defined by

$$E(h) = \begin{cases} \max_{n \geq 1, j \geq 0} |U_j^n - u(a_j, t^n)|, & \text{for population density,} \\ \max_{n \geq 1} |P^n - p(t^n)|, & \text{for total population.} \end{cases}$$

Throughout Examples 4.1–4.3, we test the scheme (3.2)–(3.5), call it *CRK*, and compare with the characteristic method (*CM*). Here, *CM* is the backward Euler method along the characteristic line $dt/da = 1$.

EXAMPLE 4.1. We solve problem (1.1) with the following data: With $a_{\dagger} = 1$,

$$m(a) = 1/(1 - a), \quad M(a, p) = p, \quad \beta(a, p) = 4,$$

and $u(a, 0) = 4(1 - a) \exp(-\alpha^* a)$. Here α^* is given by the relation $\alpha^* = \int_0^1 (\beta - m(a)) \omega(a) da$ and is computed as $\alpha^* \approx 2.5569290855$.

EXAMPLE 4.2. We solve problem (1.1) with the same data as Example 4.1 except

$$M(a, p) = p^2.$$

EXAMPLE 4.3. We solve problem (1.1) with the same data as Example 4.1 except

$$m(a) = 1/(1 - a)^3.$$

In all the examples the compatibility conditions given in Theorem 2.2 are satisfied. Tables 1–3 show the error, effective order of convergence, and *CPU* user time of both methods for the population density u and total population p when $T = 1$. We see that the processing times for the fourth order method (*CRK*) are less than twice those for the first order method (*CM*). It is also noticed from the values on these tables that the fourth order method (*CRK*) using a time step as large as $1/8$, is more accurate than the first order method (*CM*) used with $1/64$ and running times in the ratio of 1:8. We also note that (*CM*) requires a small h to have an accurate solution, as seen in the

tables. It was observed that, in actual numerical computation, nonnegative solutions were obtained for any reasonable size h , for example, $0 < h < 1$. It was also noted that the explicit Runge-Kutta methods produced a big spurious oscillation and nonnegative solution was not obtained even with small h . Thus, it is necessary to use an implicit Runge Kutta method for the initialization procedure.

h	<i>CM</i>					<i>CRK</i>				
	<i>u</i>		<i>p</i>		<i>CPU</i>	<i>u</i>		<i>p</i>		<i>CPU</i>
	$E(h)$	$r(h)$	$E(h)$	$r(h)$		$E(h)$	$r(h)$	$E(h)$	$r(h)$	
1/8	1.2745403	1.50	0.9686661	1.23	0.02	0.0062474	3.97	0.0015618	3.97	0.04
1/16	0.4479079	1.29	0.4111602	1.10	0.03	0.0003967	3.97	0.0000991	3.97	0.07
1/32	0.1823034	1.16	0.1910174	1.05	0.09	0.0000252	4.01	0.0000063	4.01	0.14
1/64	0.0811751	1.08	0.0922244	1.02	0.28	0.0000015	3.99	0.0000003	3.99	0.17

Table 1. Convergence estimates for Example 4.1.

h	<i>CM</i>					<i>CRK</i>				
	<i>u</i>		<i>p</i>		<i>CPU</i>	<i>u</i>		<i>p</i>		<i>CPU</i>
	$E(h)$	$r(h)$	$E(h)$	$r(h)$		$E(h)$	$r(h)$	$E(h)$	$r(h)$	
1/8	0.2240272	0.26	0.3623881	1.06	0.01	0.0401347	5.49	0.0100336	5.49	0.05
1/16	0.1860695	0.73	0.1636953	1.14	0.05	0.0008910	3.91	0.0002227	3.91	0.07
1/32	0.1121036	0.87	0.0784453	1.02	0.12	0.0000590	3.95	0.0000147	3.95	0.19
1/64	0.0609742	0.94	0.0384400	1.01	0.43	0.0000038	4.08	0.0000009	4.08	0.69

Table 2. Convergence estimates for Example 4.2.

h	<i>CM</i>					<i>CRK</i>				
	<i>u</i>		<i>p</i>		<i>CPU</i>	<i>u</i>		<i>p</i>		<i>CPU</i>
	$E(h)$	$r(h)$	$E(h)$	$r(h)$		$E(h)$	$r(h)$	$E(h)$	$r(h)$	
1/8	0.4149254	2.23	0.4859703	1.25	0.03	0.0074230	3.81	0.0006217	2.23	0.04
1/16	0.0880554	1.93	0.2029186	1.11	0.05	0.0005272	4.01	0.0001318	4.01	0.05
1/32	0.0230866	0.95	0.0935122	1.05	0.08	0.0000326	4.06	0.0000081	4.06	0.15
1/64	0.0119399	0.97	0.0449645	1.02	0.31	0.0000019	4.18	0.0000004	4.18	0.63

Table 3. Convergence estimates for Example 4.3.

5. Discontinuous Galerkin Method

In this section, we assume μ to be of the form

$$\mu = \lambda/(a_{\dagger} - a)^{\alpha}, 0 < \alpha < 1, \text{ for some } \lambda > 0, \text{ near the maximum age } a_{\dagger},$$

which means that the growth rate is mildly stiff. For demographic purposes, an appropriate μ can be achieved by considering a large value of λ . In that case, the solution is not smooth enough as the classical numerical schemes require and thus any finite difference scheme may not be applicable. However, if initial data is smooth enough, then the regularity of the solution is violated only along the characteristic line $t = a$ due to the compatibility condition at the origin and also at the maximum age $a_{\dagger}$ [8]. In the rest of the domain, the solution is a real analytic function. Such variations in the smoothness of the solution can be captured by using a discontinuous Galerkin method based on discontinuous piecewise polynomials. The method admits local variation of the degree of the approximating polynomial. We include the mesh points of $a = t^n$ for time level t^n and of the maximum age $a_{\dagger}$ that is the right end point of the age interval. We assume that the solution to system (1.1) is left continuous on each subinterval where the discrete solution is a polynomial. We use broken Sobolev spaces in the analysis. In order to treat the term involving the unbounded mortality function, we consider a locally defined weight function and an L^2 projection. Using them we obtain a lower bound of the bilinear form $B(u^h, \tilde{\phi})$ given in (5.7) below. The regularity of the solution depends on the growth rate of μ near the maximum age. We show that the solution belongs to L^2 independently of the growth rate of μ . We then show that the scheme is convergent, in $L^{\infty}(L^2)$ norm, at the rate of $r + 1/2$ away from the nonsmooth point and that it is convergent at the rate of $l + \alpha/2$ in $L^2(L^2)$ norm, near the singularity, if $u \in L^2(W^{l,\infty})$, where $1/2 \leq l \leq r + 1$ and r is the degree of the polynomial of the approximation space. We prove that the discrete solution is nonnegative, which is biologically meaningful. Using the property of the nonnegativity, we prove the stability of the scheme.

In order to keep the presentation simple, we leave the time variable to be continuous and we introduce a discontinuous Galerkin method to approximate the solution to (1.1). Let $V = L^2(0, a_{\dagger}]$ be the standard L^2

space and for any real numbers s and q , $1 \leq q \leq \infty$, let $\|\cdot\|_{H^s(I_m)}$ and $\|\cdot\|_{W^{s,q}(I_m)}$ be respectively, the standard norms in the standard Sobolev spaces $H^s(I_m)$ and $W^{s,q}(I_m)$ on the subinterval $I_m = (a_{m-1}, a_m]$ of $(0, a_+]$. Let $f|_S$ denote the restriction of the function f to a subset S of $(0, a_+]$. We shall suppress the explicit dependence of each function on the age variable and time in the rest of paper. Throughout the paper we will use the notation $u^h(0) := u^h(0, t)$. Furthermore, for the analysis of our method, we use the following notations:

$$\begin{aligned}
(w, v)_m &= \int_{a_{m-1}}^{a_m} w v da \quad \text{and} \quad (w, v) = \sum_{m=1}^M (w, v)_m, \\
\langle w, v \rangle_{\partial I_m} &= \langle w^-, v^- \rangle_{\partial I_m^+} + \langle w^+, v^+ \rangle_{\partial I_m^-}, \\
\langle w^-, v^- \rangle_{\partial I_m^+} &= w^-(a_m) v^-(a_m), \\
\langle w^+, v^+ \rangle_{\partial I_m^-} &= w^+(a_{m-1}) v^+(a_{m-1}), \\
\langle w^-, v^+ \rangle_{\partial I_m^-} &= w^-(a_{m-1}) v^+(a_{m-1}), \\
[w](a_m) &= w^+(a_m) - w^-(a_m), \\
\|\cdot\|_m &= (\cdot, \cdot)_m^{1/2}, \quad \|\cdot\| = \sum_{m=1}^M \|\cdot\|_m, \\
\|w\|_{\partial I_m} &= \langle w, w \rangle_{\partial I_m}^{1/2}, \\
\|w\|_{\partial I_m^\pm} &= \langle w^\mp, w^\mp \rangle_{\partial I_m^\pm}^{1/2}, \\
\|w\|_{L^q(I_m)} &= \left(\int_{a_{m-1}}^{a_m} w^q da \right)^{1/q}.
\end{aligned}$$

We now let $\Omega = (0, a_+] = \cup_{m=1}^M I_m$, $I_m = (a_{m-1}, a_m]$ and let

$$(5.1) \quad \mathcal{C}^h := \{I_m\}_{m=1}^M, \quad a_* = a_L < a_M = a_+,$$

be a regular family of subintervals of Ω indexed by a parameter h , such that $h = \max_{1 \leq m \leq M} h_m$, $h_m = a_m - a_{m-1}$. We further assume that the partitions are quasi-uniform in the sense that

$$(5.2) \quad \text{there exists a } \kappa > 0 \text{ such that } h/h_{\min} < \kappa, \quad h_{\min} = \min_{1 \leq m \leq M} h_m.$$

Given a positive integer r , we consider the following family of subspaces of V associated to it:

$$(5.3) \quad V^h = \{p : (0, a_+] \rightarrow R \mid p|_{I_m} \text{ is a polynomial of degree } \leq r\}.$$

Now, let $\partial I_m^- = a_{m-1}$ and $\partial I_m^+ = a_m$ denote the inflow and the outflow points of I_m , respectively and let $\Gamma_- = \{0\}$ and $\Gamma_+ = \{a_+\}$ be the inflow and the outflow boundaries of Ω , respectively. Then, the semi-discrete finite element method we shall analyze is given as follows: For each $1 \leq m \leq M$, for given u_m^{h-} , find $u_m^h := u^h|_{I_m} : [0, T] \rightarrow V^h|_{I_m}$,

$$(5.4) \quad \begin{aligned} & \left(\frac{\partial u_m^h}{\partial t}, v \right)_m + \left(\frac{\partial u_m^h}{\partial a} + \mu u_m^h, v \right)_m + \langle u_m^{h+}, v^+ \rangle_{\partial I_m^-} \\ & = \langle u_m^{h-}, v^+ \rangle_{\partial I_m^-}, \quad \forall v \in V^h|_{I_m}, \\ & u_m^h(a, 0) = P^h|_{I_m} u_0(a), \\ & u_1^{h-}(0) = \int_0^{a_+} \beta(a) u^h(a, t) da, \end{aligned}$$

where $u^h = \sum_{m=1}^M u^h|_{I_m}$, $P^h|_{I_m} u_0$ is the L^2 projection of u_0 onto $V^h|_{I_m}$, $v^\pm(a_m) = \lim_{\varepsilon \rightarrow 0^\pm} v(a_m \pm \varepsilon)$, and

$$(5.5) \quad V^h|_{I_m} = \{p : I_m \rightarrow R \mid p \text{ is a polynomial of degree } \leq r\}.$$

We observe that $u^h(a, \cdot)$ is left continuous and $u^{h-}(a_m, \cdot) \equiv u^h(a_m, \cdot)$, for $0 \leq m \leq M$.

If we add (5.4)₁ for $m = 1, \dots, M$, we obtain the following compact form of discontinuous Galerkin method:

$$(5.6) \quad \begin{aligned} & \left(\frac{\partial u^h}{\partial t}, v \right) + B(u^h, v) = (\beta(a), u^h) v^+(0), \quad \forall v \in V^h, \\ & u^h(a, 0) = P^h u_0(a), \end{aligned}$$

where $P^h u_0$ is the L^2 projection of u_0 onto V^h and

$$(5.7) \quad B(w, v) = \sum_{m=1}^M \left(\frac{\partial w}{\partial a} + \mu w, v \right)_m + \sum_{m=2}^M \langle [w], v^+ \rangle_{\partial I_m^-} + \langle w^+, v^+ \rangle_{\Gamma_-}.$$

Introducing basis functions for $V^h|_{I_m}$ and writing u_m^h as linear combinations of them with time dependent coefficients and extending it to u^h on the whole domain $(0, a_+]$, (5.6) is transformed into an initial value problem for a system of ODE's for the coefficients of the form:

$$(5.8) \quad A \frac{dw}{dt} = F(w),$$

where A is a positive definite mass matrix and $F(w)$ is a differentiable function and thus the general theorem for system of ODE's guarantees the unique solvability of (5.8) for small time. Due to the nonlocal birth process (1.1)₂, the coefficient matrices of system (5.8) are block diagonal and/or block Leslie matrices and each block is an $(r+1) \times (r+1)$ matrix. The system is easily solvable by block back substitution.

6. L^2 error analysis

For $\chi \in L^2(0, a_+]$, let $\tilde{\chi}$ denote the L^2 projection of χ onto V^h given by

$$(\chi - \tilde{\chi}, v)_m = 0, \quad \forall v \in V^h|_{I_m}.$$

Then, the L^2 projection $\tilde{\chi}$ has the following approximation properties [?]: For $2 \leq q \leq \infty$,

$$\begin{aligned} \|\chi - \tilde{\chi}\|_m &\leq Ch_m^{r+1} |\chi|_{H^{r+1}(I_m)}, \\ \|\chi - \tilde{\chi}\|_{\partial I_m} &\leq Ch_m^{r+1/2} |\chi|_{H^{r+1}(I_m)}, \\ \|\chi - \tilde{\chi}\|_{L^q(I_m)} &\leq Ch_m^l \|\chi\|_{W^{l,q}(I_m)}, \quad 0 \leq l \leq r+1, \\ &\quad \text{if } \chi \in V \cap W^{l,q}(I_m), \\ \|\chi - \tilde{\chi}\|_{L^q(\partial I_m)} &\leq Ch_m^{l-1/2} \|\chi\|_{W^{l,q}(I_m)}, \quad 1/2 \leq l \leq r+1, \\ &\quad \text{if } \chi \in V \cap W^{l,q}(I_m). \end{aligned}$$

Throughout the paper, C will denote a generic positive constant which is independent of h and not necessarily the same at each occurrence. We now show that the discrete solution u^h is nonnegative, which is biologically meaningful. Later, we will use the nonnegativity of u^h to prove the boundedness of u^h in L^2 and the stability of the scheme. For analysis, we rewrite the system in matrix form.

THEOREM 6.1. *The solution of (5.4), u^h , is nonnegative.*

Using Theorem 6.1, we show that the solution u^h to (5.4) is bounded in L^2 norm, which will be used in the proof of the stability of the scheme.

THEOREM 6.2. *Let u^h be the solution to (5.4). Then there exists $C > 0$ such that*

$$\|u^h\|_{L^\infty(0,T;L^2(0,a_+))} \leq C\|u_0\|_{L^2(0,a_+)},$$

$$\|\sqrt{\mu}u^h\|_{L^2(0,T;L^2(0,a_+))} \leq C\|u_0\|_{L^2(0,a_+)}.$$

We now define

$$(6.1) \quad \psi(a) = e^{-\gamma a}, \text{ where } \gamma = \min\left\{\frac{1}{\lambda}, \frac{1}{\mu}\right\}, \quad \gamma \in (0, 1].$$

We then see that $\|\psi\|_{L^\infty(0,a_+)} = 1$ and that $|\psi| \geq a_+e^{-\gamma a_+} > 0$. Consider

$$(6.2) \quad \varphi = \begin{cases} \psi u^h, & a \in [0, a_{M-1}], \\ u^h, & a \in (a_{M-1}, a_M]. \end{cases}$$

For the proof of the stability of the scheme and the error analysis, we need a lower bound of $B(u^h, \tilde{\varphi})$.

LEMMA 6.3. *Let $\mathcal{C}^h$ be a quasi-uniform family of subintervals of $(0, a_+]$ and let V^h be defined by (5.3) for some given $r \geq 0$. If u^h is the solution to (5.4) and $\tilde{\varphi}$ is given by (6.2), Then there exists positive constant $h_0 = h_0(\kappa, r)$ such that if $0 < h < h_0$, then for some $C > 0$, $B(u^h, \tilde{\varphi})$ satisfies the following:*

$$B(u^h, \tilde{\varphi}) \geq \frac{C}{2} \left[\gamma \sum_{m=1}^{M-1} \|u^h\|_m^2 + \sum_{m=1}^M \|\sqrt{\mu}u^h\|_m^2 + \sum_{m=2}^M \| [u^h] \|_{\partial I_m^-}^2 + \|u^{h+}\|_{\Gamma_-}^2 + \|u^{h-}\|_{\Gamma_+}^2 \right].$$

Using the boundedness of u^h in L^2 , we now prove the stability of the scheme:

LEMMA 6.4. *Let $\mathcal{C}^h$ be a quasi-uniform family of subintervals of $(0, a_+]$, and let V^h be defined by (5.3) for some given $r \geq 0$. Then there is a constant $h_0 = h_0(\lambda, \kappa, r)$ such that if $0 < h < h_0$, the solution u^h to (5.4) satisfies the following estimate:*

$$\begin{aligned} & \|u^h\|_{L^\infty(0,T;L^2(0,a_+])} + \sum_{m=1}^M \|\sqrt{\mu}u^h\|_{L^2(0,T;L^2(I_m))} \\ & + \sum_{m=2}^M \| [u^h] \|_{L^2(0,T;L^2(\partial I_m^-))} + \|u^{h+}\|_{L^2(0,T;L^2(\Gamma_-))} + \|u^{h-}\|_{L^2(0,T;L^2(\Gamma_+))} \\ & \leq C \|u_0\|_{L^2(0,a_+]} . \end{aligned}$$

We are now ready to prove the convergence estimate. Let $e = u^h - u$, $\theta = u^h - \tilde{u}$, $-\rho = \tilde{u} - u$. Then the error $e = u^h - u$ satisfies the following bound:

THEOREM 6.5. *Assume u belongs to $L^2(0, T; H^{r+1} \cap W^{l,\infty}(I_m))$, $1 \leq m \leq M-1$ and to $L^2(0, T; W^{l,\infty}(I_M))$ and assume that $\frac{\partial u}{\partial t}$ belongs to $L^2(0, T; H^{r+1}(I_m))$, $1 \leq m \leq M-1$ and to $L^2(0, T; W^{l,\infty}(I_M))$, for some $1/2 \leq l \leq r+1$. Let the approximate solution u^h be defined by (5.4). Then, for sufficiently small h , there exists a positive constant $C = C(a_+, \kappa, \lambda)$ satisfying the following:*

$$\begin{aligned} & \|e\|_{L^\infty(0,T;L^2(0,a_+])} + \sum_{m=1}^L \|\sqrt{\mu}e\|_{L^2(0,T;L^2(I_m))} \\ & + \sum_{m=L+1}^{M-1} \|e\|_{L^2(0,T;L^2(I_m))} + h_M^{-\alpha/2} \|e\|_{L^2(0,T;L^2(I_M))} \\ & + \sum_{m=2}^M \| [e] \|_{L^2(0,T;L^2(\partial I_m^-))} + \|e^+\|_{L^2(0,T;L^2(\Gamma_-))} + \|e^-\|_{L^2(0,T;L^2(\Gamma_+))} \\ & \leq C \left[\sum_{m=1}^M h_m^{r+1} \|u_0\|_{H^{r+1}(I_m)} + \sum_{m=1}^{M-1} h_m^{r+1/2} \|u\|_{L^2(0,T;H^{r+1}(I_m))} \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{m=L+1}^{M-1} h_m^{l+(1-\alpha)/2} \|u\|_{L^2(0,T;W^{l,\infty}(I_m))} + h_M^l \|u\|_{L^2(0,T;W^{l,\infty}(I_M))} \\
& + \sum_{m=1}^{M-1} h_m^{r+1} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T;H^{r+1}(I_m))} + h_M^l \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T;H^l(I_M))} \Big].
\end{aligned}$$

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