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GENERALIZED BOUNDED ANALYTIC FUNCTIONS IN THE SPACE $H_{\omega,p}$

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Abstract. We define a general space $H_{\omega,p}$ of the Hardy space and improve that Aleman's results to the space $H_{\omega,p}$. It follows that the multiplication operator on this space is cellular indecomposable and that each invariant subspace contains nontrivial bounded functions.

1. Introduction

For a positive integrable function $\omega \in C^2[0,1)$, we define the space $H_{\omega,p}$ of analytic functions f in the unit disc U that satisfies

$$\|f\|_{\omega,p}^p = |f(0)|^p + p^2 \int_U |f(z)|^{p-2}|f'(z)|^2\omega(|z|)dm < \infty, \tag{1.1}$$

where m is the area measure on C. Some simple computations with power series show that if $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic in U, then

$$\|f\|_{\omega,p}^p = \sum_{n\geq 0} |a_n|^p \omega_n, \tag{1.2}$$

where $\omega_0 = 1$ and for $n \geq 1$,

$$\omega_n = 2\pi n^p \int_0^1 r^{pn-p+1}\omega(r)dr. \tag{1.3}$$

If $p = 2$, Aleman([1]) proved that every function in $H_{\omega,2}$ is the quotient of two bounded analytic functions in $H_{\omega,2}$. In this paper, by the similar proofs, we shall prove that every function in $H_{\omega,p}$ is the quotient of two bounded analytic functions in $H_{\omega,p}$ for $p \geq 2$. This result which is proved

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in section 2 has some applications concerning the invariant subspaces of the multiplication operator defined on $H_{\omega,P}$ by

$$(M_z f)(\zeta) = \zeta f(\zeta), \qquad \zeta \in U, \quad f \in H_{\omega,p}. \tag{1.4}$$

Using (1.1) and (1.2), it follows easily that M_z is a bounded weighted shift on $H_{\omega,p}$. From the result mentioned above it turns out that every nontrivial invariant subspace of M_z contains a nontrivial bounded function, that each two nontrivial invariant subspaces have a nontrivial intersection and that each nontrivial invariant subspace has the codimension one property. For the usual Dirichlet space D, this was proved by S.Richter and A.Shields in [4]. These results provide positive answers to the corresponding questions for the space D_α [4, Conjectures 1 and 2] and are proved in section 3. The method used for the proofs implies the following cyclicity theorem for the spaces $H_{\omega,p}$, related to Question 3 in [2]. A function whose modulus is greater or equal to the modulus of a cyclic vector for M_z must also be a cyclic vector.

2. Bounded Functions in $H_{\omega,p}$

We begin with a general change of variable formula that is used in order to obtain an equivalent form of the norm on $H_{\omega,p}$.

LEMMA 2.1. *([1])Let ϕ be a nonconstant analytic function in U and u, v be nonnegative measurable functions on C with respect to area measure. Then*

$$\int_U (u \circ \phi) v |\phi'|^2 dm = \int_{\phi(U)} u(\zeta) \left(\sum_{\phi(z)=\zeta} v(z) \right) dm(\zeta). \tag{2.1}$$

This result is actually known and was proved for $v(z) = -log|z|$ in [5].

For a nonconstant analytic function f in U, $\zeta \in f(U)$ and u a nonnegative measurable function on $[0,1)$, we denote

$$N_{u,f}(\zeta) = \sum_{f(z)=\zeta} u(|z|). \tag{2.2}$$

In the special case when $u(r) = u_0(r) = log1/r, r \in [0,1)$, (2.2) gives the usual Nevanlinna counting function of f and we denote $N_{u_0,f} = N_f$. Substituting that $u(\zeta) = |\zeta|^{p-2}$,$v(z) = \omega(|z|)$, and $\phi(z) = f(z)$,we obtain the following Corollary.

COROLLARY 2.2. *If $f \in H_{\omega,p}$ is nonconstant, then*

$$\|f\|_{\omega,p}^p = |f(0)|^p + p^2 \int_{f(U)} |\zeta|^{p-2} \left(\sum_{f(z)=\zeta} \omega(|z|) \right) dm(\zeta).$$

$$= |f(0)|^p + p^2 \int_{f(U)} |\zeta|^{p-2} N_{\omega,f}(\zeta) dm(\zeta). \tag{2.3}$$

LEMMA 2.3. *([1])Let f be nonconstant analytic function in U and for $z, \lambda \in U$ and let $\varphi_z(\lambda) = (z+\lambda)/(1+\bar{z}\lambda)$. Then for every $\zeta \in f(U)$*

$$N_{\omega,f}(\zeta) = -\frac{1}{2\pi} \int_U \triangle\bar{\omega}(z) N_{f\circ\varphi_z}(\zeta) dm(z), \tag{2.4}$$

where $\bar{\omega}$ is defined on U by $\bar{\omega}(z) = \omega(|z|)$ and $\triangle$ denotes the Laplace operator.

From the fact that $H_{\omega,p}$ is contained in H^2, it follows that each function $f \in H_{\omega,p}$ has nontangential limits $f(e^{i\theta})a.e.$ on $[0, 2\pi]$ and that its boundary function is in $L^2[0, 2\pi]$. For $z \in U$, let $P_z(\theta) = Re(e^{i\theta}+z)/(e^{i\theta}-z)$, be the Poisson kernel.

PROPOSITION 2.4. *Let $f \in H_{\omega,p}$, then*

$$\|f\|_{\omega,p}^p - |f(0)|^p = -\int_U \triangle\bar{\omega}(z) \left(\frac{1}{2\pi} \int_0^{2\pi} P_z(\theta) |f(e^{i\theta})|^p d\theta - |f(z)|^p \right) dm(z). \tag{2.5}$$

Proof. For a nonconstant $f \in H_{\omega,p}$, we have

$$\|f\|_{\omega,p}^p - |f(0)|^p = p^2 \int_{f(U)} |\zeta|^{p-2} \left(-\frac{1}{2\pi} \int_U \triangle\bar{\omega}(z) N_{f\circ\varphi_z}(\zeta) dm(z) \right) dm(\zeta)$$

$$= -\int_U \triangle\bar{\omega}(z) \left(\frac{p^2}{2\pi} \int_{f(U)} |\zeta|^{p-2} N_{f\circ\varphi_z}(\zeta) dm(\zeta) \right) dm(z), \tag{2.6}$$

by Lemma 2.3 and Fubini's theorem. Letting $\nu(\lambda) = log 1/|\lambda|, u(\zeta) = |\zeta|^{p-2}$, $\phi = f \circ \varphi_z$ in (2.1), the Littlewood-Paley formula gives

$$\frac{p^2}{2\pi} \int_{f(U)} |\zeta|^{p-2} N_{f\circ\varphi_z}(\zeta) dm(\zeta) = \frac{p^2}{2\pi} \int_{f(U)} |\zeta|^{p-2} \left(\sum_{f\circ\varphi_z(\lambda)=\zeta} log \frac{1}{|\lambda|} \right) dm(\zeta)$$

$$= \frac{p^2}{2\pi}\int_U |(f\circ\varphi_z)|^{p-2}|(f\circ\varphi_z)'|^2 log\frac{1}{|\lambda|}dm$$

$$= \|f\circ\varphi_z\|_{H^p}^p - |f\circ\varphi_z(0)|^p. \tag{2.7}$$

Obviously $f\circ\varphi_z(0) = f(z)$ and by elementary computations with harmonic measures for the unit disk, we obtain

$$\|f\circ\varphi_z\|_{H^p}^p - |f\circ\varphi_z(0)|^p = \frac{1}{2\pi}\int_0^{2\pi} |f\circ\varphi_z(e^{i\theta})|^p d\theta - |f(z)|^p$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^p d\theta - |f(z)|^p, \tag{2.8}$$

and the proof is complete.

□

By the similar proofs of [1, Corollary 2.5], we obtain the following.

Corollary 2.5. *If $f \in H_{\omega,p}$ and F is its outer factor, then $F \in H_{\omega,p}$ and*

$$\|F\|_{\omega,p}^p - |F(0)|^p \le \|f\|_{\omega,p}^p - |f(0)|^p. \tag{2.9}$$

For a function f in the Nevanlinna class $N, f \neq 0$, we denote by ϕ_f the outer function satisfying $|\phi_f(e^{i\theta})| = min\{1, 1/|f(e^{i\theta})|\}$ a.e. on $[0, 2\pi]$; that is

$$\phi_f(z) = exp\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} log\, min\{1, 1/|f(e^{i\theta})|\}d\theta. \tag{2.10}$$

Our main result is

Theorem 2.6. *Let $f \in H_{\omega,p}, f \neq 0$. Then $\phi_f, f\phi_f$ are in $H_{\omega,p}$ and satisfy*

$$\|\phi_f\|_{\omega,p}^p - |\phi_f(0)|^p \le \|f\|_{\omega,p}^p - |f(0)|^p \tag{2.11}$$

and

$$\|f\phi_f\|_{\omega,p} \le \|f\|_{\omega,p}. \tag{2.12}$$

The proof uses the following inequalities:

LEMMA 2.7. *([1])Let (X,μ) be a probability space and $f \in L^1(\mu)$ such that $f > 0\,\mu -$ a.e. on X and $\log f \in L^1(\mu)$. Let*

$$E(f) = \int_X f d\mu - exp \int_X log\, f d\mu. \tag{2.13}$$

Then

$$E(min\{1, f\}) \le E(f), \tag{2.14}$$

and

$$E(max\{1, f\}) \le E(f), \tag{2.15}$$

Proof of Theorem 2.6. Let $f \in H_{\omega,p}$, $f = IF$ with I inner and F outer. An application of (2.14) with $X = [0, 2\pi]$, $d\mu = (1/2\pi)P_z d\theta$, yields

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f\phi_f(e^{i\theta}|^p d\theta - |F\phi_f(z)|^p \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F\phi_f(e^{i\theta})|^p d\theta - |F\phi_f(z)|^p \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, min\{1, |F(e^{i\theta})|^p\} d\theta \\
&\quad - exp\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, log\, min\{1, |F(e^{i\theta})|^p\} d\theta\right) \\
&= E(min\{1, |F|^p\}) \\
&\le \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F(e^{i\theta})|^p d\theta - exp\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, log\, |F(e^{i\theta})|^p d\theta\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F(e^{i\theta})|^p d\theta - |F(z)|^p.
\end{aligned}$$

Hence

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f\phi_f(e^{i\theta})|^p d\theta - \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\,|F(e^{i\theta})|^p d\theta \\
&\le |F\phi_f(z)|^p - |F(z)|^p = -|F(z)|^p(1 - |\phi_f(z)|^p) \\
&\le -|I(z)F(z)|^p(1 - |\phi_f(z)|^p) = |f\phi_f(z)|^p - |f(z)|^p. \qquad (2.16)
\end{aligned}$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f\phi_f(e^{i\theta})|^p d\theta - |f\phi_f(z)|^p \le \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^p d\theta - |f(z)|^p, \tag{2.17}$$

and the inequality in (2.12) follows by Proposition 2.4. Furthermore, we have

$$\left|\frac{1}{\phi_f(e^{i\theta})}\right| = max\{1, |f(e^{i\theta})|\}\, a.e.\, on\, [0, 2\pi]$$

and for $z \in U$,

$$\left|\frac{1}{\phi_f(e^{i\theta})}\right|^p = exp\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, log\, max\{1, |f(e^{i\theta})|^p\} d\theta\right). \tag{2.18}$$

We apply (2.15) to obtain

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\left|\frac{1}{\phi_f(e^{i\theta})}\right|^p d\theta - \left|\frac{1}{\phi_f(z)}\right|^p &= E(max\{1, |f|^p\}) \le E(|f|^p) \\ &\le \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^p d\theta - |f(z)|^p, \qquad (2.19) \end{aligned}$$

for all $z \in U$. Thus by Proposition 2.4, $1/\phi_f \in H_{\omega,p}$ and

$$\|\frac{1}{\phi_f}\|^p_{\omega,p} - |\frac{1}{\phi_f(0)}|^p \le \|f\|^p_{\omega,p} - |f(0)|^p. \tag{2.20}$$

Finally, since $\phi_f' = -\phi_f^2(1/\phi_f)'$ and $|\phi_f| \le 1$ in U, the definition (1.1) implies that

$$\begin{aligned} \left\|\frac{1}{\phi_f}\right\|^p_{\omega,p} - \left|\frac{1}{\phi_f(0)}\right|^p &= p^2\int_U \left|\frac{1}{\phi_f(z)}\right|^{p-2}\left|\left(\frac{1}{\phi_f(z)}\right)'\right|^2 \omega(|z|)dm \\ &\ge p^2\int_U |\phi_f(z)|^{p-2}|\phi_f'|^2\omega(|z|)dm \\ &= \|\phi_f\|^p_{\omega,p} - |\phi_f(0)|^p \qquad (2.21) \end{aligned}$$

for $p \ge 2$. Hence we obtain

$$\|\phi_f\|^p_{\omega,p} - |\phi_f(0)|^p \le \|f\|^p_{\omega,p} - |f(0)|^p,$$

and the proof is complete.

Corollary 2.8. *If $p \ge 2$, then every function in $H_{\omega,p}$ is the quotient of two bounded functions in $H_{\omega,p}$.*

Proof. Since $|\phi_f|$, $|f\phi_f| \leq 1$ in U and $f = f\phi_f/\phi_f$, we have the result. $\square$

3. Invariant subspaces

The present section contains some applications of Theorem 2.6 concerning the invariant subspaces of the multiplication operator on $H_{\omega,p}$ defined by (1.4). A closed subspace M of $H_{\omega,p}$ is called invariant if $M_z M \subset M$. For a function $f \in H_{\omega,p}$ we denote by $[f]$ the smallest invariant subspace containing f ; that is the closure of the polynomial multiplies of f in $H_{\omega,p}$. In order to prove the main result of this section we use the same method as in [4]. Let H^∞ be the algebra of bounded analytic functions in U with the norm $\|g\|_\infty = sup_{z\in U}|g(z)|$, $g \in H^\infty$. We have

LEMMA 3.1. *If $f \in H_{\omega,p}$ and $g \in H^\infty$ such that $gf \in H_{\omega,p}$, then $gf \in [f]$.*

The proof of the lemma is based on some simple properties of the linear operators T_i, $0 \leq t < 1$, defined on the set $H(U)$ of analytic functions in U by

$$(T_i h)(z) = \frac{1}{1-t}\int_t^1 h(sz)ds \;\; z \in U, \;\; h \in H(U). \tag{3.1}$$

By the theorem of L'hospital, we have

$$lim_{t->1}(T_i h)(z) = lim_{t->1}\frac{\int_1^t h(sz)ds}{t-1} = lim_{t->1}h(tz) = h(z)$$

for all $z \in U$ and $h \in H(U)$. Some other properties are summarized below. For $h \in H(U)$, and $t \in [0,1)$, we denote by h_t the function given by $h_t(z) = h(tz)$, $z \in U$.

LEMMA 3.2. *For every $t \in [0,1)$, we have*

$$(i)(M_z T_t h)' = \frac{h - th_t}{1-t}, \;\; h \in H(U).$$

$$(ii) If \;\; h \in H_{\omega,p}\,, \; then \; T_t h \in H_{\omega,p} \; and \; \|T_t h\|_{\omega,p} \leq \|h\|_{\omega,p}\,.$$

(iii)If $h \in H^\infty$, then $T_t h \in H_{\omega,p}$ and $fT_t h \in H_{\omega,p}$ whenever $f \in H_{\omega,p}$. Furthermore in this case, $fT_t h \in [f]$.

Proof. (i)For every $\zeta \in U$ and $h \in H(U)$,
(3.2)
$$(M_z T_t h)(\zeta) = \zeta (T_t h)(\zeta) = \zeta \frac{1}{1-t} \int_t^1 h(s\zeta) ds = \frac{1}{1-t} \int_{t\zeta}^{\zeta} h(\lambda) d\lambda.$$

Hence

$$\begin{aligned}(M_z T_t h)'(\zeta) &= \left[\frac{1}{1-t} \int_0^{\zeta} h(\lambda) d\lambda - \frac{1}{1-t} \int_0^{t\zeta} h(\lambda) d\lambda \right]' \\ &= \frac{1}{1-t} \left[h(\zeta) - t h_t(\zeta) \right]\end{aligned}$$

(ii)If $h(z) = \sum_{n \geq 0} a_n z^n$, $z \in U$, then

$$(T_t h)(z) = \sum_{n \geq 0} \left(\frac{1 - t^{n+1}}{1-t} \right) \frac{a_n}{n+1} z^n, \quad z \in U, \tag{3.3}$$

which shows that $T_t h \in H_{\omega,p}$ whenever $h \in H_{\omega,p}$ and $\|T_t h\|_{\omega,p} \leq \|h\|_{\omega,p}$.
(iii)From (i) we obtain that if $h \in H^\infty$ then $T_t h$ and $(T_t h)'$ are both in H^∞,hence $T_t h \in H_{\omega,p}$ and also a multiplier. Moreover, there exists a sequence of polynomials $\{p_n\}$ with $sup_n \|p_n'\|_\infty < \infty$, conversing pointwise to $T_t h$ on U. It follows that $sup_n \|p_n f\|_\omega < \infty$, and that at least a subsequence of $\{p_n f\}$ converges weakly in $H_{\omega,p}$. Also, its limit must be $f T_t h$ because the point evaluations are bounded linear functionals on $H_{\omega,p}$. Thus, $f T_t h \in [f]$. □

Proof of Lemma 3.1. Let f, g be as in the statement. For every $t \in [0,1)$ we have $f T_t g \in [f]$ by Lemma 3.2 and $lim_{t->1} (f T_t g)(z) = f(z) g(z)$ for all $z \in U$. We are going to show that the norms $\|f T_t g\|_{\omega,p}$ remain bounded when t tends to 1. Note first that by the definition of $H_{\omega,p}$, it follows easily that the operator M_z is injective and has closed range on $H_{\omega,p}$, hence there exists a positive constant c such that $\|f T_t g\| \leq c \|M_z f T_t g\|_{\omega,p}$, $t \in [0,1)$. Further,

$$\begin{aligned}\|M_z f T_t g\|_{\omega,p}^p &\leq 2 \int_U |f (M_z T_t g)'|^p \omega \, dm + 2 \int_U |f' M_z T_t g|' \omega \, dm \\ &\leq 2 \int_U |f (M_z T_t g)'|^p \omega \, dm + 2 \|g\|_\infty^p \|f\|_\omega^p,\end{aligned} \tag{3.4}$$

and by Lemma 3.2(i),
(3.5)
$$|f(M_zT_tg)'| = \left|f\frac{g-tg_t}{1-t}\right| = \left|\frac{fg-tf_tg-t-g_tf+g_ttf_t+fg_t(1-t)}{1-t}\right|$$
$$\leq \left|\frac{fg-tf_tg_t}{1-t}\right| + \left|g_t\frac{f-tf_t}{1-t}\right| + |fg_t|$$
$$= |(M_zT_tfg)'| + |g_t(M_zT_tf)'| + |fg_t|.$$
We obtain
$$\int_U |f(M_zT_tg)'|^p\omega\, dm \leq 3\|M_zT_tfg\|^p_{\omega,p} + 3\|g\|^p_{\omega,p}\|M_zT_tf\|^p_{\omega,p} + 3\|g\|^p_{\omega,p}\int_U |f|^p\omega\, dm. \tag{3.6}$$
This leads to an estimation of the form
$$\|fT_tg\|^p_{\omega,p} \leq c_1\|fg\|^p_{\omega,p} + c_2\|g\|^p_{\omega,p}\|f\|^p_{\omega,p}, \tag{3.7}$$
where c_1, c_2 are positive constants independent of t. As in the proof of Lemma 3.2(iii), there exists a sequence $\{t_n\}$ in $[0,1)$, tending to 1 such that $fT_{t_n}g$ tends weakly to fg in $H_{\omega,p}$, hence $gf \in [f]$.

A function $f \in H_{\omega,p}$ is called a cyclic vector for M_z if $[f] = H_{\omega,p}$. From Lemma 3.1, we obtain

Corollary 3.3. *If f, $g \in H_{\omega,p}$, g is a cyclic vector for M_z and $|f(z)| \geq |g(z)|$ for all $z \in U$, then f is also cyclic.*

Proof. If $h = g/f$, then $h \in H^\infty$ and $hf = g$,i.e. $hf \in H_{\omega,p}$. Then by Lemma 3.1, $g \in [f]$, which shows that f is cyclic. □

Remark. The above result remains true if $H_{\omega,p}$ is replaced by the Dirichlet space D, because Lemma 3.1 holds for this space as well, with the same proof. This problem was raised for general Banach spaces of analytic functions by L. Brown and A. Shields [2, Question 3].

The main result of this section is

Theorem 3.4. *Let $M \neq \{0\}$, $M \neq \{0\}$ be invariant subspaces for the operator M_z on $H_{\omega,p}$. Then (i) $M \cap H^\infty \neq \{0\}$ and (ii) $M \cap N \neq \{0\}$.*

Proof. (i)Let $f \in M$, $f =\neq 0$. By Theorem 2.6 there exist functions $g, h \in H^\infty \cap H_{\omega,p}$ such that $f = g/h$ and, by Lemma 3.1, $g = hf \in [f] \subset M$. (ii) If $g \in M$, $h \in N$, $g, h \neq 0$ are bounded then $gh \in [g] \cap [h] \subset M \cap N$. □

Theorem 3.4(ii) states that the operator M_z on $H_{\omega,p}$ is cellular indecomposable and answers affirmatively Conjecture 1 of [4]. As it was pointed out in [4] this implies the fact that each nontrivial invariant subspace M of M_z has the codimension one property that is, $(z - \lambda)M$ is a closed subspace of M having codimension 1 in M, for every $\lambda \in U$. This follows from results obtained by S. Richter in [3] and Theorem 3.4. Indeed, if M is such a subspace and $f, g \in M \backslash \{0\}$, then $[f] \cap [g] \neq \{0\}$ by Theorem 3.4 and by [3, Corollaries 3.12 and 3.15] M has the codimension one property.

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