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FUZZY K-PROXIMITY MAPPING

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Abstract. This paper is devoted to the study of the role of fuzzy proximity spaces. We define a fuzzy K-proximally continuous mapping based on the fuzzy K-proximity and prove some of its properties.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [8] in 1965. This idea was used by Chang [1], who in 1968 defined fuzzy topological spaces, and by Lowen [4], who in 1974 defined fuzzy uniform spaces. Furthermore, Katsaras [2], who in 1979, defined fuzzy proximities, on the base of the axioms suggested by Efremovič[6].

In this paper we try to characterize the fuzzy K-proximally continuous based on the fuzzy K-proximity.

2. Preliminaries

As a preparation, we briefly review some basic definitions concerning a fuzzy proximity space. Throughout this paper, X is reserved to denote a nonempty set and let I^X be the collection of all mappings from X to the unit closed interval $I = [0, 1]$ of the real line. A member μ of I^X is called a fuzzy set of X. For any $\mu, \rho \in I^X$, the join $\mu \vee \rho$, and the meet $\mu \wedge \rho$ of μ and ρ defined as followings: For any $x \in X$, $(\mu \vee \rho)(x) = \sup\{\mu(x), \rho(x)\}$ and $(\mu \wedge \rho)(x) = \inf\{\mu(x), \rho(x)\}$, respectively. And $\mu \leq \rho$ if for each $x \in X$, $\mu(x) \leq \rho(x)$. The complement μ^c of a fuzzy set μ in X is $1 - \mu$ defined by $\mu^c(x) = (1 - \mu)(x) = 1 - \mu(x)$ for

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each $x \in X$. 0 and 1 denote constant mappings all of X to 0 and 1, respectively. Now we give the definition of a fuzzy topology.

Definition 2.1. A fuzzy topology on X is a subset α of I^X which satisfies the following conditions:

(1) (FT1) $0, 1 \in \alpha$.
(2) (FT2) If $\mu, \rho \in \alpha$, then $\mu \wedge \rho \in \alpha$.
(3) (FT3) If $\mu_i \in \alpha$ for each $i \in A$, then $\sup_{i \in A} \mu_i \in \alpha$.

The pair (X, α) is called a fuzzy topological space, of its for short.

In the following we first define a fuzzy proximity space and a fuzzy point. Let δ be a relation on I^X, i.e., $\delta \subset I^X \times I^X$. The facts that $(\mu, \rho) \in \delta$ and $(\mu, \rho) \notin \delta$ are denoted by $\mu\delta\rho$ and $\mu\bar{\delta}\rho$, respectively.

Definition 2.2. A relation δ on I^X is called a fuzzy proximity if δ satisfies the following conditions :

(1) (FP1) $\mu\delta\rho$ implies $\rho\delta\mu$.
(2) (FP2) $(\mu \vee \rho)\delta\sigma$ if and only if $\mu\delta\sigma$ or $\rho\delta\sigma$.
(3) (FP3) $\mu\delta\rho$ implies $\mu \neq 0$ and $\rho \neq 0$.
(4) (FP4) $\mu\bar{\delta}\rho$ implies that there exists a $\rho \in I^X$ such that $\mu\bar{\delta}\sigma$ and $(1-\sigma)\bar{\delta}\rho$.
(5) (FP5) $\mu \wedge \rho \neq 0$ implies $\mu\delta\rho$.

The pair (X, δ) is called a fuzzy proximity space.

Definition 2.3. A fuzzy set in X is called a fuzzy point if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\gamma (0 < \gamma < 1)$, we denote this fuzzy point by x_γ, where the point x is called its support.

Definition 2.4. The fuzzy point x_γ is said to be contained in a fuzzy set μ, or to belong to μ, denoted by $x_\gamma \in \mu$, if $\gamma < \mu(x)$. Evidently, every fuzzy set μ can be expressed as the union of all the fuzzy points which belong to μ.

We introduce a fuzzy K-proximity.

Definition 2.5. A relation δ on I^X is called a fuzzy K-proximity if δ satisfies the following conditions:

(1) (FK1) $x_\gamma\delta(\mu \vee \rho)$ if and only if $x_\gamma\delta\mu$ or $x_\gamma\delta\rho$.
(2) (FK2) $x_\gamma\bar{\delta}0$ for all x_γ.

(3) (FK3) $x_\gamma \in \mu$ implies $x_\gamma \delta \mu$.
(4) (FK4) $x_\gamma \overline{\delta} \mu$ implies that there exists a $\rho \in I^X$ such that $x_\gamma \overline{\delta} \rho$ and $y_\gamma \overline{\delta} \mu$ for all $y_\gamma \in (1-\rho)$.

The pair (X, δ) is called a fuzzy K-proximity space.

One can easily show that the fuzzy proximity on I^X implies the fuzzy K-proximity on I^X.

Now we shall introduce the fuzzy proximity δ_1 from the fuzzy K-proximity δ replacing the axiom (FK4) in the fuzzy K-proximity by the stronger one.

Definition 2.6. A relation δ on I^X is called the fuzzy proximity if δ satisfies the axioms (FP1), (FP2), (FP3) in Definition 2.2, and the modified axiom(FP4$'$)For each $\sigma \in I^X$ there is a fuzzy point x_γ such that either $x_\gamma \delta \mu$, $x_\gamma \delta \sigma$ or $x_\gamma \delta \rho$, $x_\gamma \delta (1-\sigma)$, then we have $x_\gamma \delta \mu$ and $x_\gamma \delta \rho$.

Definition 2.7. In a fuzzy K-proximity space (X, δ), let $\delta_1$be a relation on I^X defined as follows: For each $\mu, \rho \in I^X$, $\mu \delta_1 \rho$ if and only if there is a fuzzy point x_γ such that $x_\gamma \delta \mu$ and $x_\gamma \delta \rho$.

Given a fuzzy proximity space (X, δ), ρ may said to be a fuzzy proximity neighborhood of μ if and only if $\mu \overline{\delta} (1-\rho)$ for $\mu, \rho \in I^X$. An analogous concept, that of a fuzzy K-proximity neighborhood, can be introduced in a fuzzy K-proximity space.

Definition 2.8. Let (X, δ) be a fuzzy K-proximity space. For $\mu \in I^X$, we say that μ is a $\delta-$neighborhood of a fuzzy point x_γ (in symbols $x_\gamma \ll \mu$) if $x_\gamma \overline{\delta} (1-\mu)$.

3. Fuzzy K-proximally continuous mapping

In the study of general topological spaces, continuous mappings play an important role. A similar role is played by uniformly continuous mappings in uniform space. Their analogue in the theory of fuzzy K-proximity spaces is the concept of fuzzy K-proximity (or fuzzy K-proximally continuous) mapping.

Definition 3.1. Let (X, δ_1) and (Y, δ_2) be two fuzzy K-proximity spaces. A mapping $f : X \to Y$ is said to be a fuzzy K-proximity mapping if $x_\gamma \delta_1 \mu$ implies $f(x_\gamma)\delta_2 f(\mu)$. Equivalently, f is a fuzzy K-proximity mapping if $x_\gamma \overline{\delta_2} \mu$ implies $f^{-1}(x_\gamma)\overline{\delta_1} f^{-1}(\mu)$ or $x_\gamma \ll_2 \mu$ implies $f^{-1}(x_\gamma) \ll_1 f^{-1}(\mu)$.

It is easy to see that the composition of two fuzzy K-proximity mappings is a fuzzy K-proximity mapping.

The next theorem is similar to the well-known result: a uniformly continuous mapping is continuous with respect to the induced topologies.

THEOREM 3.1. *A fuzzy K-proximity mapping $f : (X, \delta_1) \to (Y, \delta_2)$ is continuous with respect to $\mathcal{T}(\delta_1)$ and $\mathcal{T}(\delta_2)$.*

Proof. The result of the theorem follows from the fact that $x_\gamma \delta_1 \mu$ implies $f(x_\gamma)\delta_2 f(\mu)$. i.e. $f(\overline{\mu}) \subset \overline{f(\mu)}$. □

REMARK. The converse of Theorem 3.1. is false. Consider the identity mapping on X is continuous with respect to $\mathcal{T}(\delta_1)$ and $\mathcal{T}(\delta_2)$, but is not a fuzzy K-proximity mapping from (X, δ_1) to (X, δ_2).

THEOREM 3.2. *Given a mapping $f : X \to (Y, \delta_2)$, the coarsest fuzzy K-proximity δ_0 which may be assigned to X in order that f be fuzzy K-proximally continuous is defined by $x_\gamma \overline{\delta_0} \mu$ if and only if there exists a $\rho \in I^Y$ such that $f(x_\gamma)\overline{\delta_2}(1-\rho)$ and $f^{-1}(\rho) \in (1-\mu)$.*

Proof. We first verify that δ_0 is a fuzzy K-proximity on X.

(FK1) $x_\gamma \overline{\delta_0}(\mu \vee \rho)$ implies the existence of a $\gamma \in I^Y$ such that $f(x_\gamma)\overline{\delta_2}(1-\gamma)$ and $f^{-1}(\gamma) \in (1-(\mu \vee \rho))$, from which $x_\gamma \overline{\delta_0} \mu$ and $x_\gamma \overline{\delta_0} \rho$ follow. If $x_\gamma \overline{\delta_0} \mu$ and $x_\gamma \overline{\delta_0} \rho$, there exist γ_1 and γ_2 such that $f(x_\gamma)\overline{\delta_2}(1-\gamma_1)$, $f(x_\gamma)\overline{\delta_2}(1-\gamma_2)$, $f^{-1}(\gamma_1) \in (1-\mu)$ and $f^{-1}(\gamma_2) \in (1-\rho)$. Therefore, $f(x_\gamma)\overline{\delta_2}(1-(\gamma_1 \vee \gamma_2))$ and $f^{-1}(\gamma_1 \vee \gamma_2) \in (1-(\mu \vee \rho))$, i.e. $x_\gamma \overline{\delta_0}(\mu \vee \rho)$.

(FK2) It is clear that $x_\gamma \overline{\delta_0} 0$ for all x_γ.

(FK3) If $x_\gamma \overline{\delta_0} \mu$, then there exists a $\rho \in I^Y$ such that $f(x_\gamma)\overline{\delta_2}(1-\rho)$ and $f^{-1}(\rho) \in (1-\mu)$. Therefore, $f(x_\gamma) \notin (1-\rho)$ and $f^{-1}(f(x_\gamma)) \notin f^{-1}(1-\rho)$. Since $x_\gamma \in f^{-1}(f(x_\gamma))$ and $\mu \in f^{-1}(1-\rho)$, we have $x_\gamma \notin \mu$.

(FK4) If $x_\gamma \overline{\delta_0} \mu$, then there exists a $\rho \in I^Y$ such that $f^{-1}(\rho) \in (1-\mu)$ and $f(x_\gamma)\overline{\delta_2}(1-\rho)$. This latter and (FK4) together assure the existence of a $\gamma \in I^Y$ such that $f(x_\gamma)\overline{\delta_2}\gamma$ and $y_\gamma \overline{\delta_2}(1-\rho)$ for all $y_\gamma \in (1-\gamma)$. Let $\alpha = f^{-1}(\gamma)$. Since $f(x_\gamma)\overline{\delta_2}\gamma$, we can get that $x_\gamma \overline{\delta_0}\alpha$. As $f(1-\alpha) \in (1-\gamma)\overline{\delta_2}(1-\rho)$ and $f^{-1}(\rho) \in (1-\gamma)$, we can have that $y_\gamma \overline{\delta_0}\mu$ for all $y_\gamma \in (1-\alpha)$.

In order to show that $f : (X, \delta_0) \to (Y, \delta_2)$ is fuzzy K-proximally continuous, suppose that $f(x_\gamma)\overline{\delta_2} f(\mu)$. Since $f(x_\gamma) \ll (1-f(\mu))$, there exists a $\rho \in I^Y$ such that $f(x_\gamma) \ll \rho \ll (1-f(\mu))$. Thus $f(x_\gamma)\overline{\delta_2}(1-\rho)$ and $f^{-1}(\rho) \in (1-\mu)$, i.e. $x_\gamma \overline{\delta_0}\mu$.

It remains to show that if δ_1 is any fuzzy K-proximity on X such that $f : (X, \delta_1) \to (Y, \delta_2)$ is fuzzy K-proximally continuous, then δ_1 is finer than δ_0. If $x_\gamma \overline{\delta_0}\mu$, then there exists a $\rho \in I^Y$ such that $f(x_\gamma)\overline{\delta_2}(1-\rho)$ and $f^{-1}(\rho) \in (1-\mu)$. Since f is fuzzy k-proximally continuous, $x_\gamma \overline{\delta_1}(1-f^{-1}(\rho))$, and $\mu \in (1-f^{-1}(\rho))$ implies $x_\gamma \overline{\delta_1}\mu$. Thus $\delta_1 > \delta_0$. □

References

1. C.L. Chang, *Fuzzy topological space*, J. Math. Anal.Appl. **24** (1968), 182–190.
2. A.K. Katsaras, *Fuzzy proximity spaces*, J. Math. Anal. Appl. **68** (1979), 100–110.
3. C.Y. Kim, K.L. Choi and Y.S. Shin, *On the K-proximities*, Kyungpook Mathematical Journal **13(1)** (1973), 21–32.
4. R. Lowen, *Fuzzy uniform spaces*, J. Math. Anal. Appl. **82** (1981), 370–385.
5. K.D. Park, *On the Fuzzy K-proximities*, J. Nat.Sci. Res. Inst. Dongguk Univ. **14** (1994), 19–24.
6. P-M. Pu and Y-M. Liu, *Fuzzy topology 1*, J. Math. Anal. Appl. **76** (1980), 571–599.
7. S.A. Naimpally and B.D. Warrack, *Proximity spaces*, Cambridge Univ. Press, New York (1970).
8. C.K. Wong, *Fuzzy points and local properties of fuzzy topology*, J. Math. Anal. Appl. **46** (1974), 316–328.
9. L.A. Zadeh, *Fuzzy sets*, Informs. Contr. **8** (1965), 333–353.

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