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SOME RESULTS ON AN INTUITIONISTIC FUZZY TOPOLOGICAL SPACE

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Abstract. In this paper, we introduce the concepts of r-closure and r-interior defined by intuitionistic gradation of openness. We also introduce the concepts of r-gp-maps, weakly r-gp-maps, and obtain some characterizations in terms of r-closure and r-interior operators.

1. Introduction

In 1992 [8], Chattopadyay et. al. introduced the concept of fuzzy topology redefined by a gradation of openness and investigated some fundamental properties. In particular, Gayyar, Kerre, Ramadan [7] and Demirci [5, 6] introduced the concepts of fuzzy closure and fuzzy interior of a fuzzy set, and obtained some properties of them. Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of fuzzy set in Zadeh's sense [11]. Çoker introduced the concept of intuitionistic fuzzy topological spaces [4], which it is an extended concept of fuzzy topological spaces [2] in Chang's sense. In 2002, Mondal and Samanta introduced and investigated the concept of intuitionistic gradation of openness [9] which is a generalization of the concept of gradation of openness defined by Chattopadyay.

In this paper, we introduce the concepts of r-closure and r-interior defined by intuitionistic gradation of openness. We also introduce the concepts of weakly r-gp-maps, r-gp-maps, weakly r-gp-maps, and obtain some characterizations.

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2. Preliminaries

Let X be a set and $I = [0, 1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

Definition 2.1 ([3, 8, 10]). Let X be a non-empty set and $\tau : I^X \to I$ be a mapping satisfying the following conditions:
1. $\tau(0_X) = \tau(1_X) = 1$.
2. $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$.
3. For every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J}\ \tau(A_i)$.

Then the mapping $\tau : I^X \to I$ is called a fuzzy topology (or gradation of openness [10]) on X. We call the ordered pair (X, τ) a fuzzy topological space. The value $\tau(A)$ is called the degree of openness of A.

Definition 2.2 ([1]). An intuitionistic fuzzy set A in a set X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Definition 2.3 ([9]). An intuitionistic gradation of openness (briefly IGO) of fuzzy subsets of a set X is an ordered pair (τ, τ^*) of functions $\tau, \tau^* : I^X \to I$ such that
(IGO1) $\tau(A) + \tau^*(A) \leq 1$, for all $A \in I^X$,
(IGO2) $\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0$,
(IGO3) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B)$,
(IGO4) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J}\ \tau(A_i)$ and $\tau^*(\cup_{i\in J} A_i) \leq \vee_{i\in J}\ \tau^*(A_i)$.

Then the triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space (briefly $IFTS$) on X. τ and τ^* may be interpreted as gradation of openness and gradation of non-openness, respectively.

Definition 2.4 ([9]). Let X be a nonempty set and two functions $\mathcal{F}, \mathcal{F}^* : I^X \to I$ be satisfying

(IGC1) $\mathcal{F}(A) + \mathcal{F}^*(A) \leq 1$, for all $A \in I^X$,
(IGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = 1, \mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = 0$,
(IGC3) $\forall A, B \in I^X$, $\mathcal{F}(A \cup B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$ and $\mathcal{F}^*(A \cup B) \leq \mathcal{F}^*(A) \vee \mathcal{F}^*(B)$,
(IGC4) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\mathcal{F}(\cap_{i \in J} A_i) \geq \wedge_{i \in J} \mathcal{F}(A_i)$ and $\mathcal{F}^*(\cap_{i \in J} A_i) \leq \vee_{i \in J} \mathcal{F}^*(A_i)$.

Then the ordered pair $(\mathcal{F}, \mathcal{F}^*)$ is called an intuitionistic gradation of closedness [9] (briefly *IGC*) on X. $\mathcal{F}$ and $\mathcal{F}^*$ may be interpreted as gradation of closedness and gradation of nonclosedness, respectively.

THEOREM 2.5 ([9]). *Let X be a nonempty set. If (τ, τ^*) is an IGO on X, then the pair $(\mathcal{F}, \mathcal{F}^*)$, defined by $\mathcal{F}_\tau(A) = \tau(A^c)$, ${\mathcal{F}^*}_{\tau^*}(A) = \tau^*(A^c)$ where A^c denotes the complement of A, is an IGC on X. And if $(\mathcal{F}, \mathcal{F}^*)$ is an IGC on X, then the pair $(\tau_\mathcal{F}, {\tau^*}_{\mathcal{F}^*})$, defined by $\tau_\mathcal{F}(A) = \mathcal{F}(A^c)$, ${\tau^*}_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c)$ is an IGO on X.*

DEFINITION 2.6 ([9]). Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a *gp-map* if $\tau(f^{-1}(A)) \geq \sigma(A)$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$ for every $A \in I^Y$.

3. r-Closure and r-Interior Operators in IFTS

In this section, we introduce the concepts of r-closure and r-interior of a fuzzy set on IFTS and investigate some their properties.

DEFINITION 3.1. Let (X, τ, τ^*) be an IFTS, $A \in I^X$ and $r \in [0, 1)$. Then the r-closure (resp., r-interior) of A, denoted by $cl_r A$ (resp., $i_r A$), is defined by $cl_r A = \cap\{K \in I^X : \mathcal{F}_\tau(K) > 0$ and ${\mathcal{F}^*}_{\tau^*}(K) \leq r, A \subseteq K\}$ (resp., $i_r A = \cup\{K \in I^X : \tau(K) > 0$ and $\tau^*(K) \leq r, K \subseteq A\}$).

THEOREM 3.2. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$, $r \in [0, 1)$. Then*

1. *$i_r A \subset A \subset cl_r A$.*
2. *If $A \subseteq B$, then $cl_r A \subseteq cl_r B$ and $i_r A \subseteq i_r B$.*
3. *$(i_r A)^c = cl_r A^c$.*
4. *$(cl_r A)^c = i_r(A^c)$.*

Proof. (1) and (2) follow directly from Definition 3.1.

(3) From Theorem 2.5 and Definition 3.1, we have that

$$\begin{aligned}(cl_r A)^c &= (\cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*{}_{\tau^*}(K) \le r, A \subseteq K\})^c \\ &= \cup\{K^c : K \in I^X, \tau(K^c) > 0 \text{ and } \tau^*(K^c) \le r, K^c \subseteq A^c\} \\ &= \cup\{U \in I^X : \tau(U) > 0 \text{ and } \tau^*(U) \le r, U \subseteq A^c\} \\ &= i_r(A^c).\end{aligned}$$

The proof of (4) is similar to the proof of (3). □

THEOREM 3.3. *Let (X, τ, τ^*) be an IFTS and $A \in I^X$, $r \in [0, 1)$. Then*

1. $\tau(A) > 0$ *and* $\tau^*(A) \le r \Rightarrow i_r A = A$.
2. $\mathcal{F}_\tau(A) > 0$ *and* $\mathcal{F}^*{}_{\tau^*}(A) \le r \Rightarrow cl_r A = A$.

Proof. (1) Let $\tau(A) > 0$ and $\tau^*(A) \le r$. Then $A \in \{K \in I^X : \tau(K) > 0 \text{ and } \tau^*(K) \le r, K \subseteq A\}$. By Definition 3.1 and Theorem 3.2, it follows $i_r A = A$.

(3) Let $\mathcal{F}_\tau(A) > 0$ and $\mathcal{F}^*{}_{\tau^*}(A) \le r$. Then $A \in \{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*{}_{\tau^*}(K) \le r, A \subseteq K\}$. Thus by Definition 3.1 and Theorem 3.2, we get $cl_r A = A$.

□

EXAMPLE 3.4. Let $X = I$ and let N denote the set of all natural numbers. Consider a fuzzy set $\mu_n \in I^X$ for each $n \in N$ such that $\mu_n(x) = \frac{n-1}{n}x$ for $x \in X$.

Define an intuitionistic gradation of openness $\tau, \tau^* : I^X \to I$ by

$$\begin{aligned}&\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0, \\ &\tau(\mu_n) = \frac{1}{n}, \tau^*(\mu_n) = \frac{n-1}{2n}, \\ &\tau(\mu) = 0, \tau^*(\mu) = \frac{1}{2} \text{ for all other fuzzy set } \mu \in I^X.\end{aligned}$$

Consider a fuzzy set $A \in I^X$ such that $A(x) = x$ for all $x \in X$ and $r = \frac{2}{3}$. Then it follows $i_r A = A$ but $\tau(A) = 0,\ \tau^*(A) = \frac{1}{2}$.

Thus the converse of the part (1) in Theorem 3.3 is not true in general.

THEOREM 3.5. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$, $r \in [0, 1)$. Then*

1. $cl_r(0_X) = 0_X$.

2. $A \subseteq cl_r A$.
3. $cl_r A = cl_r(cl_r A)$.
4. $cl_r A \cup cl_r B \subseteq cl_r(A \cup B)$.

Proof. (1) and (2) follow from Definition 3.1 and Theorem 3.2.

(3) By Definition 3.1, for $A \in I^X$ we can write that
$cl_r A = \cap\{H \in I^X : \mathcal{F}_\tau(H) > 0 \text{ and } \mathcal{F}^*{}_{\tau^*}(H) \leq r, A \subseteq H\}$.
But since $\mathcal{F}_\tau(H) > 0$ and $\mathcal{F}^*{}_{\tau^*}(H) \leq r$, by Theorem 3.2 and Theorem 3.3 we get $A \subseteq cl_r A \subseteq cl_r H = H$. Thus

$$\begin{aligned} cl_r A &= \cap \{H \in I^X : \mathcal{F}_\tau(H) > 0 \text{ and } \mathcal{F}^*{}_{\tau^*}(H) \leq r, A \subseteq H\} \\ &\supseteq \cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*{}_{\tau^*}(K) \leq r, cl_r A \subseteq K\} \\ &= cl_r(cl_r A). \end{aligned}$$

Consequently we have $cl_r(cl_r A) = cl_r A$ from (2).

(4) It follows from (2). □

Theorem 3.6. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$, $r \in [0, 1)$. Then*

1. $i_r(1_X) = 1_X$.
2. $i_r A \subseteq A$.
3. $i_r(i_r A) = i_r A$.
4. $i_r(A \cap B) \subseteq i_r A \cap i_r B$.

Proof. The proof is similar to the proof of Theorem 3.5.

□

Definition 3.7. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs, and $r \in [0, 1)$. A mapping $f : X \to Y$ is a *r-gp-map* iff $\sigma(A) \leq \tau(f^{-1}(A))$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$, for each a fuzzy set A in Y such that $\sigma(A) > 0$ and $\sigma^*(A) \leq r$.

Definition 3.8. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs, and $r \in [0, 1)$. A mapping $f : X \to Y$ is a *weakly r-gp-map* iff $\tau(f^{-1}(A)) > 0$ and $\tau^*(f^{-1}(A)) \leq r$, for each fuzzy set $A \in I^Y$ such that $\sigma(A) > 0$ and $\sigma^*(A) \leq r$.

It is obvious that every weakly r-gp-map is a r-gp-map from the above definitions. But we can show that the converse is not always true from the following example:

EXAMPLE 3.9. Let $X = I$ and let N denote the set of all natural numbers. For each $n \in N$, we consider $\mu_n \in I^X$ such that $\mu_n(x) = \frac{1}{n}x$ for $x \in X$.

Define $\tau, \tau^* : I^X \to I$ by

$$\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0;$$
$$\tau(\mu_n) = \frac{n}{n+2}, \tau^*(\mu_n) = \frac{2}{n+2} \text{ for each } n \in N;$$
$$\tau(\mu) = 0, \tau^*(\mu) = 1 \text{ for all other fuzzy set } \mu \in I^X.$$

And define $\sigma, \sigma^* : I^X \to I$ by

$$\sigma(0_X) = \sigma(1_X) = 1, \sigma^*(0_X) = \sigma^*(1_X) = 0;$$
$$\sigma(\mu_n) = \frac{1}{n+1}, \sigma^*(\mu_n) = \frac{1}{n+1} \text{ for each n in } N;$$
$$\sigma(\mu) = 0, \sigma^*(\mu_n) = 1 \text{ for all other fuzzy set } \mu \in I^X.$$

Then the pairs (τ, τ^*) and (σ, σ^*) are two intuitionistic gradations of openness on X.

Consider the identity mapping $f : (X, \tau, \tau^*) \to (Y, \sigma, \sigma^*)$ and $r = \frac{1}{2}$. Then f is a weakly $\frac{1}{2}$-gp-map but not a $\frac{1}{2}$-gp-map. For if $2 \leq n$, then $\sigma(\mu_n) \leq \tau(\mu_n)$ but $\tau^*(\mu_n) \nleq \sigma^*(\mu_n)$.

THEOREM 3.10. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs, and $r \in [0, 1)$. A mapping $f : X \to Y$ is a r-gp-map iff $\mathcal{F}_\sigma(A) \leq \mathcal{F}_\tau(f^{-1}(A))$ and $\mathcal{F}^*{}_{\tau^*}(f^{-1}(A)) \leq \mathcal{F}^*{}_{\sigma^*}(A)$, for each $A \in I^Y$ such that $\mathcal{F}_\sigma(A) > 0$ and $\mathcal{F}^*{}_{\sigma^*}(A) \leq r$.*

Proof. Suppose that f is a r-gp-map and let $\mathcal{F}_\sigma(A) > 0$ and $\mathcal{F}^*{}_{\sigma^*}(A) \leq r$ for $A \in I^Y$; then $\mathcal{F}_\sigma((A^c)^c) = \sigma(A^c) > 0$. Since f is a r-gp-map, it follows $\sigma(A^c) \leq \tau(f^{-1}(A^c))$ and $\tau^*(f^{-1}(A^c)) \leq \sigma^*(A^c)$. Thus from Theorem 2.5, we get $\mathcal{F}_\sigma(A) \leq \mathcal{F}_\tau(f^{-1}(A))$ and $\mathcal{F}^*{}_{\tau^*}(f^{-1}(A)) \leq \mathcal{F}^*{}_{\sigma^*}(A)$. The converse is obvious.

□

THEOREM 3.11. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs, $r \in [0, 1)$. A mapping $f : X \to Y$ is a weakly r-gp-map iff $\mathcal{F}_\tau(f^{-1}(A)) > 0$ and $\mathcal{F}^*{}_{\tau^*}(f^{-1}(A)) \leq r$, for each fuzzy set A in Y such that $\mathcal{F}_\sigma(A) > 0$ and $\mathcal{F}^*{}_{\sigma^*}(A) \leq r$.*

Proof. It is similar to Theorem 3.10.

□

THEOREM 3.12. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs, $r \in [0, 1)$. If a mapping $f : X \to Y$ is a weakly r-gp-map, then we have*

1. $f(cl_r A) \subseteq cl_r f(A)$ *for every* $A \in I^X$,
2. $cl_r(f^{-1}(A)) \subseteq f^{-1}(cl_r A)$ *for every* $A \in I^Y$,
3. $f^{-1}(i_r A) \subseteq i_r(f^{-1}(A))$ *for every* $A \in I^Y$.

Proof. (1) Let $A \in I^X$; then by Definition 3.1 and Theorem 3.11, we have

$$\begin{aligned} f^{-1}(cl_r f(A)) &= f^{-1}[\cap\{U \in I^Y : \mathcal{F}_\sigma(U) > 0 \text{ and } \mathcal{F^*}_{\sigma^*}(U) \le r, f(A) \subseteq U\}] \\ &\supseteq \cap\{f^{-1}(U) \in I^X : \mathcal{F}_\tau(f^{-1}(U)) > 0 \text{ and } \mathcal{F^*}_{\tau^*}(f^{-1}(U)) \le r, A \subseteq f^{-1}(U)\} \\ &\supseteq cl_r A. \end{aligned}$$

Consequently, we get $f(cl_r A) \subseteq cl_r f(A)$.
(2) It follows from (1).

(3) It obtains by (2) and Theorem 3.2. □

COROLLARY 3.13. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs, $r \in [0, 1)$. If a mapping $f : X \to Y$ is a r-gp-map, then we have

1. $f(cl_r A) \subseteq cl_r f(A)$ for every $A \in I^X$,
2. $cl_r(f^{-1}(A)) \subseteq f^{-1}(cl_r A)$ for every $A \in I^Y$,
3. $f^{-1}(i_r A) \subseteq i_r(f^{-1}(A))$ for every $A \in I^Y$.

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