Kangweon-Kyungki Math. Jour. **14** (2006), No. 1, pp. 79–83

ZERMELO'S NAVIGATION PROBLEM ON HERMITIAN MANIFOLDS

Nany Lee

Abstract. In this paper, we apply Zermelo's problem of navigation on Riemannian manifolds to Hermitian manifolds. Using a similar technique with which we define a Randers metric in a Finsler manifold by perturbing Riemannian metric with a vector field, we construct an (a, b, f)-metric in a Rizza manifold from a Hermitian metric and a given vector field.

1. Introduction

In [BRS04], Bao, Robles and Shen dealt with Zermelo's problem of navigation on Riemannian manifolds and Randers metric as its solution. Here we will consider Zermelo's navigation problem on Hermitian manifolds.

Let M be a smooth $2n$-dimensional manifold with almost complex structure f and a Riemannian metric h which is compatible with f. Let W be a vector field on M. As in Zermelo's problem of navigation on Riemannian manifolds, W can be considered as a force of a wind or a current. But this time, we will think that W accompanies another influential force fW. So we have a combined force $W + fW$.

In [BRS04], if $h(W + fW, W + fW) < 1$, i.e., $h(W, W) < 1/2$, then we have a Randers metric L from the data of the Riemannian metric h and the vector field $W + fW$. We show that the necessary and sufficient condition for this Randers metric L to be a Rizza metric is that W must be a zero vector field. So we need to modify Randers metric L by adding one correction term in order to be a Rizza metric. The resulting metric

Received April 8, 2006.
2000 Mathematics Subject Classification: Primary 53B40; Secondary 53C60.
Key words and phrases: Finsler metric, generalized Randers metric, (a, b, f)-metric, Rizza manifold, Zermelo's navigation problem.
Partially supported by the University of Seoul, 2004.

happens to be an (a, b, f)-metric, which is an example of a generalized Randers metric.

In [Lee03], we computed the fundamental tensor g_{ij} of (a, b, f)-metric and its inverse g^{ij}. Here we prove that (a, b, f)-metric is a Rizza metric by showing that $g_{ij} f^i_k f^k y^j = 0$. As for Randers metric in the Finsler manifolds, (a, b, f)-metrics are very interesting class in Rizza manifolds. For further description of (a, b, f)-metrics, we refer to [I-H95, I-H96] and [Lee03].

2. Preliminaries

Let (M, f, L) be a $2n$-dimensional manifold with an almost complex structure f and a Finsler metric L. In [Riz62, Riz63, Riz64], G. B. Rizza introduced the so-called Rizza condition

$$L(x, \phi_\theta(y)) = L(x, y) \text{ for all } x \in M\,,\ y \in T_x M \text{ and } \theta \in \mathbb{R}\,,$$

where ${\phi_\theta}^i_j = (\cos\theta)\delta^i_j + (\sin\theta) f^i_j$.

In [Heil65], E. Heil showed that if the fundamental tensor g_{ij} of the Finsler metric L satisfies $g_{pq}(x, y) f^p_i(x) f^q_j(x) = g_{ij}$, then the Finsler metric L is a priori a Riemannian metric. Thus it is necessary to consider a weak condition on the Finsler metric like the Rizza condition. Note that the Rizza condition is equivalent to $g_{pq}(x, y) f^p_k(x) y^k y^q = 0$.

Recall that a generalized Randers metric is a Finsler metric in the form $L = \alpha + \beta$, where α is a Riemannian metric and β is a singular Riemannian metric. If β is a 1-form, then L is a Randers metric.

Now we will consider generalized Randers metrics on almost Hermitian manifolds. Let M be a $2n$-dimensional Riemannian manifold with an almost complex structure f and a Riemannian metric α which is compatible with f. Given a non-vanishing covariant vector field $b_i(x)$ on M, we get a singular Riemannian metric

$$\beta(x, y) = (b_{ij}(x) y^i y^j)^{1/2}\,,$$

where $b_{ij} = b_i b_j + f_i f_j$, $f_i = b_r f^r_i$. Such $L = \alpha + \beta$ is an interesting example of a generalized Randers metric. We call this metric an (a, b, f)-metric and (M, L) an (a, b, f)-manifold.

LEMMA 2.1. *A (a, b, f)-metric $L = \alpha + \beta$ satisfies a Rizza condition.*

Proof. The fundamental tensor g_{ij} of L can be written by

$$g_{ij} = \frac{L}{\alpha}a_{ij} + \frac{L}{\beta}b_i b_j + \frac{L}{\beta}f_i f_j + L_i L_j - \frac{L}{\alpha}\alpha_i\alpha_j - \frac{L}{\beta}\beta_i\beta_j\,,$$

where $\alpha_i = \frac{\partial\alpha}{\partial y^i}\,,\ \beta_i = \frac{\partial\beta}{\partial y^i}\,,\ L_i = \alpha_i + \beta_i\,$. It is sufficient to show that $g_{pq}(x,y)f^p_k(x)y^k y^q = 0$. By direct calculation, we get

$$a_{pq}(x,y)f^p_k(x)y^k y^q = 0, \qquad \alpha_p(x,y)f^p_k(x)y^k = \frac{a_{pq}(x,y)f^p_k(x)y^k y^q}{\alpha(y)} = 0,$$

$$b_p b_q f^p_k(x)y^k y^q = -f_p f_q f^p_k(x)y^k y^q,$$

$$L_p L_q f^p_k(x)y^k y^q = L\frac{b_{pq}(x,y)f^p_k(x)y^k y^q}{\beta(y)} = \frac{L}{\beta}\beta_p\beta_q f^p_k(x)y^k y^q$$

using the fact that $f\circ f = -Id$ and $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is Hermitian. This leads to $g_{pq}(x,y)f^p_k(x)y^k y^q = 0$. □

3. Construction of (a,b,f)-metrics

Recall the following in [BRS04]:

Proposition 3.1. *A strongly convex Finsler metric is of Randers metric $L = \alpha + \beta$ if and only if L solves the Zermelo navigation problem on a Riemannian manifold (M,h), with the influence W satisfying $h(W,W) < 1$.*

$L = \alpha + \beta$ is related with the Riemannian metric h and the vector field W by the following formulas

$$\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}, \qquad \beta(x,y) = b_i(x)y^i,$$

where

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda}, \qquad b_i = -\frac{W_i}{\lambda}$$

and $W_i = h_{ij}W^j$ and $\lambda = 1 - W^i W_i$.

Now let M be a $2n$-dimensional Riemannian manifold with an almost complex structure f and consider a Riemannian metric h satisfying $h(X,Y) = h(fX,fY)$. Let W be a vector field with $h(W,W) < 1$. By perturbing a Riemannian metric h under the influence of W, we obtain Randers metric $L = \alpha + \beta$.

Suppose $L = \alpha + \beta$ is a Rizza metric, i.e., $g_{pq} f^p_k y^k y^q = 0$. The fundamental tensor g_{ij} of $L = \alpha + \beta$ is

$$g_{ij} = \frac{L}{\alpha} a_{ij} - \frac{\beta}{\alpha} l_i l_j + l_i b_j + l_j b_i + b_i b_j,$$

with $l_i = \alpha_{y^i} = \dfrac{a_{ik} y^k}{\alpha}$.

By direct calculation, we get

$$g_{pq} f^p_k y^k y^q = \frac{L}{\alpha} \left\{ a_{pq} f^p_k y^k y^q + \alpha(y)\beta(fy) \right\} = 0,$$

$$a_{pq} f^p_k y^k y^q + \alpha(y)\beta(fy) = 0.$$

Plugging -y in y , we also get

$$a_{pq} f^p_k y^k y^q - \alpha(y)\beta(fy) = 0.$$

Thus $\beta(fy) = 0$ which means $W = 0$.

Proposition 3.2. *Let (M, h) be a $2n$-dimensional Riemannian manifold with an almost complex structure f satisfying $h(X, Y) = h(fX, fY)$. Let $L = \alpha + \beta$ be a solution to the Zermelo navigation problem on the Riemannian manifold (M, h) under the influence W. Then L is also a Rizza metric if and only if $W = 0$.*

Let W be any vector field with $h(W+fW, W+fW) = 2h(W, W) < 1$. From the data of the Riemannian metric h and the vector field $W + fW$, we get the Randers metric $L_o = \alpha_o + \beta_o$. By the above argument, L_o is not a Rizza metric. So we need the correction term

$$\Delta(y) = \frac{2}{\lambda^2} h(W, y) h(fW, y).$$

Now we will construct a Rizza metric.

Theorem 3.3. *Let (M, h) be a $2n$-dimensional Riemannian manifold with an almost complex structure f satisfying $h(X, Y) = h(fX, fY)$ and W be a vector field with $h(W, W) \neq 1$. Let α and β be such that*

$$\alpha^2 = \alpha_o^2 - \Delta \quad \text{and} \quad \beta^2 = \beta_o^2 - \Delta.$$

Then $L = \alpha + \beta$ is an (a, b, f)-metric and L is regular on $\mathcal{D}$, where $\mathcal{D}$ is a complement of $\{y | h(W, y) = h(fW, y) = 0\}$.

Proof. We can have $a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda} + \frac{W_p f_i^p}{\lambda}\frac{W_q f_j^q}{\lambda}$ which satisfies $a_{pq}(x) f_k^p f_j^q = a_{ij}(x)$. If we let $b_i = \frac{W_i}{\lambda}$, then $\beta(x,y) = (b_i b_j + f_i f_j)^{1/2}$ with $f_i = b_p f_i^p$. Thus this $L = \alpha + \beta$ is a (a,b,f)-metric. By the Theorem 4.1 in [Lee03], L is strongly convex on $\mathcal{D}$. Thus $L = \alpha + \beta$ is a y-local Finsler structure on $\mathcal{D}$ and satisfies Rizza condition. □

References

[BCS00] D. Bao, S. S. Chern, and Z. Shen, *An introduction to Riemannian–Finsler geometry*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 2000.

[BRS04] D. Bao, S. S. Robles, and Z. Shen, *Almost Hermitian Finsler manifolds*, J. Diff. Geom. **66** (2004), 377–435.

[Heil65] E. Heil, *A relation between Finslerian and Hermitian metrics*, Tensor(N.S.) **16** (1965), 1–3.

[I-H96] Y. Ichiiyō, M. Hashiguchi, *On (a,b,f)-metrics II*, Rep. Fac. Sci. Kagoshima Univ. Math. Phys. Chem **29** (1996), 1–5.

[I-H95] Y. Ichiiyō, M. Hashiguchi, *On (a,b,f)-metrics*, Rep. Fac. Sci. Kagoshima Univ. Math. Phys. Chem **28** (1995), 1–9.

[Lee03] N. Lee, *On the special Finsler metric*, Bull. Korean Math. Soc. **38** (2003), no. 2, 303–315.

[Mat86] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Japan, 1986.

[Riz62] G. B. Rizza, *Strutture di Finsler sulle varietà quasi complesse*, Lincei Rend. **33** (1962), no. 8, 271–275.

[Riz63] G. B. Rizza, *Strutture di Finsler di tipo quasi Hermitiano*, Riv. Mat. Univ. Parma **4** (1963), no. 2, 83–106.

[Riz64] G. B. Rizza, *F-forme quadratiche ed hermitiane*, Rend. Mat. Appl. **23** (1964), no. 5, 221–249.

Department of Mathematics
The University of Seoul
Seoul, 130–743, Korea
E-mail: nany@uos.ac.kr