ON INTRA-REGULAR ORDERED SEMIGROUPS

D. M. Lee and S. K. Lee*

Abstract. In this paper we give some new characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups.

1. Introduction


In this paper we give some new characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups.

2. Definitions and Lemmas

An ordered semigroup (po-semigroup) is a semigroup S with an ordered relation “≤” such that for all x ∈ S a ≤ b implies xa ≤ xb and ax ≤ bx ([1]).

A po-semigroup S is called intra-regular if for every a ∈ S there exist x, y ∈ S such that a ≤ xay ([4]).
Let $S$ be a po-semigroup. A non-empty subset $A$ of $S$ is called a left (resp. right) ideal of $S$ if 1) $SA \subseteq A$ (resp. $AS \subseteq A$), 2) $a \in A, b \leq a$ for $b \in S$ implies $b \in A$. If $A$ is both a left ideal and a right ideal of $S$, then $A$ is called an ideal of $S$ ([3]).

A non-empty subset $Q$ of $S$ is called a quasi-ideal of $S$ if 1) $(QS] \cap (SQ] \subseteq Q$, 2) $a \in Q, b \leq a$ for $b \in S$ implies $b \in Q$. A non-empty subset $B$ of $S$ is called a bi-ideal of $S$ if 1) $BSB \subseteq B$, 2) $a \in B, b \leq a$ for $b \in S$ implies $b \in B$ ([4, 5, 7]).

For an ordered semigroup $S$, we denote by $R(a)$ (resp. $L(a), Q(a), B(a)$) the right (resp. left, quasi-, bi-) ideal of $S$ generated by $a(a \in S)$.

For $H \subseteq S$, we denote $(H] := \{t \in S|t \leq h \text{ for some } h \in H\}$.

We can prove easily that for a non-empty subset of a po-semigroup $S$

$$R(a) = (a \cup aS), \quad L(a) = (a \cup Sa),$$

$$Q(a) = (a \cup ((aS] \cap (Sa])], \quad B(a) = (a \cup a^2 \cup aSa].$$

The ideal of $S$ generated by $a$ is the set $(a \cup Sa \cup aS \cup SaS]$ which is equal to $R(L(a)) = L(R(a))$(see[6]).

For all other definitions we refer to [2, 4, 8].

We have the following lemma (see [3]).

**Lemma.** Let $S$ be a po-semigroup. Then we have

1) $A \subseteq (A]$ for any $A \subseteq S$.

2) If $A \subseteq B$, then $(A] \subseteq (B]$.

3) If $A$ is some types of ideal, then $A = (A]$.

4) If $A$ is a subset of $S$, then ($(A)] = (A]$.

5) $(A](B] \subseteq (AB]$ for all $A, B \subseteq S$.

6) For $A, B \subseteq S$, $(A \cap B] \neq (A] \cap (B]$, in general. In particular, if $A$ and $B$ are some types of ideal of $S$, then $(A \cap B] = (A] \cap (B]$.

### 3. Main Theorems

N. Kehayopulu, S. Lajos and M. Tsingelis gave many characterizations of the intra-regular po-semigroups([6]). Now we give some new characterizations of the intra-regular po-semigroups in terms of bi-ideals, quasi-ideals, left ideals and right ideals of po-semigroups.
THEOREM 1. Let $S$ be a po-semigroup. Then the followings are true.

(1) $S$ is intra-regular if and only if for a bi-ideal $B$ and a quasi-ideal $Q$ of $S$, we have $B \cap Q \subseteq (SBQS]$.

(2) $S$ is intra-regular if and only if for a bi-ideal $B$ and a quasi-ideal $Q$ of $S$, we have $B \cap Q \subseteq (SQBS]$.

Proof. (1) Let $a \in B \cap Q$. Since $S$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Thus $a \leq xa^2y \leq xa(xa^2y)y = (axa)ay^2 \in S(BSB)QS \subseteq SBQS$. Therefore, $B \cap Q \subseteq (SBQS]$.

Conversely, let $B(a)$ be a bi-ideal and $Q(a)$ a quasi-ideal generated by $a$ in $S$. Then by hypothesis,

$$a \in B(a) \cap Q(a) \subseteq (SB(a)Q(a))S$$

$$= (S(a \cup a^2 \cup aSa)[a \cup ((aS) \cap (Sa))]S)$$

$$\subseteq ((Sa \cup Sa^2 \cup SaSa)[a \cup (aS)][S]$$

$$\subseteq ((Sa)[a \cup (aS)][S]$$

$$\subseteq ((Sa)[aS \cup (aS^2)]$$

$$\subseteq ((Sa^2S) \cup (Sa^2S^2)]$$

$$\subseteq ((Sa^2S] = (Sa^2S].$$

Therefore, $S$ is intra-regular.

(2) Let $a \in B \cap Q$. Since $S$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Thus $a \leq xa^2y \leq x(xa^2y)ay = x^2a(aya)y \in SQ(BSB)S \subseteq SQBS$. Therefore, $B \cap Q \subseteq (SQBS]$.

Conversely, let $B(a)$ be a bi-ideal and $Q(a)$ a quasi-ideal generated by $a$ of $S$. Then by hypothesis,

$$a \in B(a) \cap Q(a) \subseteq (SQ(a)B(a))S$$

$$= (S(a \cup ((aS) \cap (Sa))) \cup a^2 \cup aSa)aS]$$

$$\subseteq (S(a \cup (Sa))[aS \cup a^2S \cup aSaS]$$

$$\subseteq ((Sa \cup (S^2a))[aS] \subseteq ((Sa)(aS]$$

$$\subseteq ((Sa^2S] = (Sa^2S].$$

Therefore, $S$ is intra-regular. □
Theorem 2. Let $S$ be a po-semigroup. Then the followings are true.  

(1) $S$ is intra-regular if and only if for a left ideal $L$ and a bi-ideal $B$ of $S$, we have $L \cap B \subseteq (LBS)$. 

(2) $S$ is intra-regular if and only if for a right ideal $R$ and a bi-ideal $B$ of $S$, we have $B \cap R \subseteq (SBR)$. 

Proof. (1) Let $a \in L \cap B$. Since $S$ is intra-regular, there exists $x, y \in S$ such that $a \leq xa^2y \leq x(xa^2)y = x^2a(aya)y \in SL(BSB)S \subseteq LBS$. Thus $L \cap B \subseteq (LBS)$.

Conversely, let $B(a)$ be a bi-ideal generating by $a$ and $L(a)$ a left ideal generating by $a$. Then by hypothesis, 

$$a \in L(a) \cap B(a) \subseteq (L(a)B(a)S] = ((a \cup Sa)[a \cup a^2 \cup aSa][a])$$

$$\subseteq ((a \cup Sa)(aS \cup a^2S \cup aSaS])$$

$$\subseteq ((a \cup Sa)(aS]) = (a^2S \cup Sa^2S].$$

Hence $a \leq t$ for some $t \in (a^2S \cup Sa^2S]$. If $t \in a^2S$, then $a \leq a^2x$ for some $x \in S$. Thus we have $a \leq a^2x \leq a(a^2x)x = aa^2x^2 \in Sa^2S$. If $t \in Sa^2S$, it is obvious. Therefore $a \in (Sa^2S]$ for any cases. Hence $S$ is intra-regular.

(2) Let $a \in B \cap R$. Since $S$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y \leq x(a^2x)y = x(axa)y^2 \in S(BSB)RS \subseteq SBR$. Thus $B \cap R \subseteq (SBR)$.

Conversely, let $B(a)$ be a bi-ideal generating by $a$ and $R(a)$ a right-ideal generating by $a$. Then by hypothesis, 

$$a \in B(a) \cap R(a) \subseteq (SB(a)R(a)] = (S[a \cup a^2 \cup aSa][a \cup aS])$$

$$\subseteq ((S[a \cup a^2 \cup aSa][a \cup aS])$$

$$\subseteq ((S[a][a \cup aS])$$

$$\subseteq ((S[a \cup a^2S]) = (Sa^2 \cup Sa^2S].$$

Hence $a \leq t$ for some $t \in (Sa^2 \cup Sa^2S]$. If $t \in Sa^2$, then $a \leq xa^2$ for some $x \in S$. Thus we have $a \leq xa^2 \leq x(a^2) = x^2a^2a \in Sa^2S$. If
On intra-regular ordered semigroups

$t \in Sa^2S$, it is obvious. Therefore $a \in (Sa^2S)$ for any cases. Hence $S$ is intra-regular. \hfill \Box

Remark. A semigroup $S$ is a po-semigroup with the relation $i = \{(a, a) | \forall a \in S\}$. Thus we obtain Theorem 1, 2, 3 and 4 in [11] from above Theorem 1 and 2.

We can prove that the following theorem 3 from Theorem 1 and 2.

Theorem 3. Let $S$ be a po-semigroup and $a \in S$. Then the followings are true.

1. $S$ is intra-regular.
2. $B(a) \cap Q(a) \subseteq (SB(a)Q(a)S]$.  
3. $B(a) \cap Q(a) \subseteq (SB(a)Q(a)S]$.  
4. $B(a) \cap Q(a) \subseteq (SQ(a)B(a)S]$.  
5. $B(a) \cap Q(a) \subseteq (SQ(a)B(a)S]$.  
6. $L(a) \cap B(a) \subseteq (L(a)B(a)S]$.  
7. $L(a) \cap B(a) \subseteq (L(a)B(a)S]$.  
8. $B(a) \cap R(a) \subseteq (SB(a)R(a)]$.  
9. $B(a) \cap R(a) \subseteq (SB(a)R(a)]$.  

References


Department of Mathematics Education and RINS  
College of Education  
Gyeongsang National University  
Jinju, 660–701, Korea  
*E-mail: sklee@gsnu.ac.kr*