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ON INTRA-REGULAR ORDERED SEMIGROUPS

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Abstract. In this paper we give some new characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups.

1. Introduction

In [9, 10] S. Lajos and G. Szasz characterized the intra-regular semigroups in terms of right ideals and left ideals of semigroups. Also, in [6] N. Kehayopulu, S. Lajos and M. Tsingelis characterized the intra-regular ordered semigroups in terms of right ideals and left ideals of ordered semigroups, and they obtained many characterizations of the intra-regularity in ordered semigroups . Recently, in [11] S. K. Lee characterized the intra-regular semigroups -without order- in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, and bi-ideals and right ideals of semigroups.

In this paper we give some new characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups.

2. Definitions and Lemmas

An *ordered semigroup* (: *po-semigroup*) is a semigroup S with an ordered relation " $\leq$ " such that for all $x \in S$ $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$([1]).

A *po*-semigroup S is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$ ([4]).

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Let S be a *po*-semigroup. A non-empty subset A of S is called a *left* (resp. *right*) *ideal* of S if 1) $SA \subseteq A$ (resp. $AS \subseteq A$), 2) $a \in A, b \leq a$ for $b \in S$ implies $b \in A$. If A is both a left ideal and a right ideal of S, then A is called an *ideal* of S ([3]).

A non-empty subset Q of S is called a *quasi-ideal* of S if 1) $(QS] \cap (SQ] \subseteq Q$, 2) $a \in Q, b \leq a$ for $b \in S$ implies $b \in Q$. A non-empty subset B of S is called a *bi-ideal* of S if 1) $BSB \subseteq B$, 2) $a \in B, b \leq a$ for $b \in S$ implies $b \in B$ ([4, 5, 7]).

For an ordered semigroup S, we denote by $R(a)$ (resp. $L(a), Q(a), B(a)$) the right (resp. left, quasi-, bi-) ideal of S generated by $a (a \in S)$.

For $H \subseteq S$, we denote $(H] := \{t \in S | t \leq h \text{ for some } h \in H\}$.

We can prove easily that for a non-empty subset of a *po*-semigroup S

$$R(a) = (a \cup aS], \quad L(a) = (a \cup Sa],$$

$$Q(a) = (a \cup ((aS] \cap (Sa])], \quad B(a) = (a \cup a^2 \cup aSa].$$

The ideal of S generated by a is the set $(a \cup Sa \cup aS \cup SaS]$ which is equal to $R(L(a)) = L(R(a))(see[6])$.

For all other definitions we refer to [2, 4, 8].

We have the following lemma (see [3]).

Lemma. *Let S be a po-semigroup. Then we have*
1) $A \subseteq (A]$ for any $A \subseteq S$.
2) If $A \subseteq B$, then $(A] \subseteq (B]$.
3) If A is some types of ideal, then $A = (A]$.
4) If A is a subset of S, then $((A]] = (A]$.
5) $(A](B] \subseteq (AB]$ for all $A, B \subseteq S$.
6) For $A, B \subseteq S$, $(A \cap B] \neq (A] \cap (B]$, in general. In particular, if A and B are some types of ideal of S, then $(A \cap B] = (A] \cap (B]$.

3. Main Theorems

N. Kehayopulu, S. Lajos and M. Tsingelis gave many characterizations of the intra-regular *po*-semigroups([6]). Now we give some new characterizations of the intra-regular *po*-semigroups in terms of bi-ideals, quasi-ideals, left ideals and right ideals of *po*-semigroups.

Theorem 1. *Let S be a po-semigroup. Then the followings are true.*

(1) S is intra-regular if and only if for a bi-ideal B and a quasi-ideal Q of S, we have $B \cap Q \subseteq (SBQS]$.

(2) S is intra-regular if and only if for a bi-ideal B and a quasi-ideal Q of S, we have $B \cap Q \subseteq (SQBS]$.

Proof. (1) Let $a \in B \cap Q$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Thus $a \leq xa^2y \leq xa(xa^2y)y = x(axa)ay^2 \in S(BSB)QS \subseteq SBQS$. Therefore, $B \cap Q \subseteq (SBQS]$.

Conversely, let $B(a)$ be a bi-ideal and $Q(a)$ a quasi-ideal generated by a in S. Then by hypothesis,

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (SB(a)Q(a)S] \\ &= (S(a \cup a^2 \cup aSa](a \cup ((aS] \cap (Sa]))]S] \\ &\subseteq ((Sa \cup Sa^2 \cup SaSa](a \cup (aS]]S] \\ &\subseteq ((Sa](a \cup (aS]]S] \\ &\subseteq ((Sa](aS \cup (aS^2]] \\ &\subseteq ((Sa^2S] \cup (Sa^2S^2]] \\ &\subseteq ((Sa^2S]] = (Sa^2S]. \end{aligned}$$

Therefore, S is intra-regular.

(2) Let $a \in B \cap Q$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Thus $a \leq xa^2y \leq x(xa^2y)ay = x^2a(aya)y \in SQ(BSB)S \subseteq SQBS$. Therefore, $B \cap Q \subseteq (SQBS]$.

Conversely, let $B(a)$ be a bi-ideal and $Q(a)$ a quasi-ideal generated by a of S. Then by hypothesis,

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (SQ(a)B(a)S] \\ &= (S(a \cup ((aS] \cap (Sa]))](a \cup a^2 \cup aSa]S] \\ &\subseteq (S(a \cup (Sa]](aS \cup a^2S \cup aSaS]] \\ &\subseteq ((Sa \cup (S^2a]](aS]] \subseteq ((Sa](aS]] \\ &\subseteq ((Sa^2S]] = (Sa^2S]. \end{aligned}$$

Therefore, S is intra-regular. $\square$

THEOREM 2. *Let S be a po-semigroup. Then the followings are true.*

(1) S is intra-regular if and only if for a left ideal L and a bi-ideal B of S, we have $L \cap B \subseteq (LBS]$.

(2) S is intra-regular if and only if for a right ideal R and a bi-ideal B of S, we have $B \cap R \subseteq (SBR]$.

Proof. (1) Let $a \in L \cap B$. Since S is intra-regular, there exists $x, y \in S$ such that $a \le xa^2y \le x(xa^2y)ay = x^2a(aya)y \in SL(BSB)S \subseteq LBS$. Thus $L \cap B \subseteq (LBS]$.

Conversely, let $B(a)$ be a bi-ideal generating by a and $L(a)$ a left ideal generating by a. Then by hypothesis,

$$\begin{aligned} a \in L(a) \cap B(a) &\subseteq (L(a)B(a)S] \\ &= ((a \cup Sa](a \cup a^2 \cup aSa]S] \\ &\subseteq ((a \cup Sa](aS \cup a^2S \cup aSaS]] \\ &\subseteq ((a \cup Sa](aS]] = (a^2S \cup Sa^2S]. \end{aligned}$$

Hence $a \le t$ for some $t \in (a^2S \cup Sa^2S]$. If $t \in a^2S$, then $a \le a^2x$ for some $x \in S$. Thus we have $a \le a^2x \le a(a^2x)x = aa^2x^2 \in Sa^2S$. If $t \in Sa^2S$, it is obvious. Therefore $a \in (Sa^2S]$ for any cases. Hence S is intra-regular.

(2) Let $a \in B \cap R$. Since S is intra-regular, there exist $x, y \in S$ such that $a \le xa^2y \le xa(xa^2y)y = x(axa)ay^2 \in S(BSB)RS \subseteq SBR$. Thus $B \cap R \subseteq (SBR]$.

Conversely, let $B(a)$ be a bi-ideal generating by a and $R(a)$ a right-ideal generating by a. Then by hypothesis,

$$\begin{aligned} a \in B(a) \cap R(a) &\subseteq (SB(a)R(a)] \\ &= (S(a \cup a^2 \cup aSa](a \cup aS]] \\ &\subseteq ((Sa \cup Sa^2 \cup SaSa](a \cup aS]] \\ &\subseteq ((Sa](a \cup aS]] \\ &\subseteq ((Sa^2 \cup Sa^2S]] = (Sa^2 \cup Sa^2S]. \end{aligned}$$

Hence $a \le t$ for some $t \in (Sa^2 \cup Sa^2S]$. If $t \in Sa^2$, then $a \le xa^2$ for some $x \in S$. Thus we have $a \le xa^2 \le x(xa^2)a = x^2a^2a \in Sa^2S$. If

$t \in Sa^2S$, it is obvious. Therefore $a \in (Sa^2S]$ for any cases. Hence S is intra-regular. □

REMARK. A semigroup S is a *po*-semigroup with the relation $i = \{(a, a) | \forall a \in S\}$. Thus we obtain Theorem 1, 2, 3 and 4 in [11] from above Theorem 1 and 2.

We can prove that the following theorem 3 from Theorem 1 and 2.

THEOREM 3. *Let S be a po-semigroup and $a \in S$. Then the followings are true.*

(1) S is intra-regular.
(2) $B(a) \cap Q(a) \subseteq (SB(a)Q(a)S]$.
(3) $B(a) \cap Q(a) \subseteq (SB(a)Q(a)S]$.
(4) $B(a) \cap Q(a) \subseteq (SQ(a)B(a)S]$.
(5) $B(a) \cap Q(a) \subseteq (SQ(a)B(a)S]$.
(6) $L(a) \cap B(a) \subseteq (L(a)B(a)S]$.
(7) $L(a) \cap B(a) \subseteq (L(a)B(a)S]$.
(8) $B(a) \cap R(a) \subseteq (SB(a)R(a)]$.
(9) $B(a) \cap R(a) \subseteq (SB(a)R(a)]$.

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