

BANACH SPACE WITH PROPERTY (β) WHICH CANNOT BE RENORMED TO BE B-CONVEX

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ABSTRACT. In this paper, we study property (β) and B-convexity in reflexive Banach spaces. It is shown that k -uniform convexity implies B-convexity and property (β) . We also show that there is a Banach space with property (β) which cannot be equivalently renormed to be B-convex.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X . By B_X and S_X , we denote the closed unit ball and the unit sphere of X , respectively. For any subset A of X by $\text{co}(A)(\overline{\text{co}}(A))$ we denote the convex hull (closed convex hull) of A

$(X, \|\cdot\|)$ is called uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta.$$

A k -uniformly convex space is defined for $k \geq 2$ in an obvious fashion so that a uniformly convex space is just 2-uniformly convex; $(X, \|\cdot\|)$ is k -uniformly convex if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x_1, x_2, \dots, x_k \in B_X$ with $\|x_i - x_j\| \geq \epsilon$ for $i \neq j$ and $i, j = 1, 2, \dots, k$,

$$\left\| \frac{1}{k} \sum_{i=1}^k x_i \right\| \leq 1 - \delta.$$

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For a sequence (x_n) in X , we let

$$\text{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\}.$$

$(X, \|\cdot\|)$ is said to have the Kadec-Klee property (KK) if for $x_n \in B_X$, $\text{sep}(x_n) > 0$ and $x_n \rightarrow x$ weakly, then $\|x\| < 1$. The followings are intermediate notions between (UC) and (KK).

$(X, \|\cdot\|)$ is uniformly Kadec-Klee property (UKK) if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x_n \in B_X$, $\text{sep}(x_n) \geq \epsilon$ and $x_n \rightarrow x$ weakly, then $\|x\| \leq 1 - \delta$.

$(X, \|\cdot\|)$ is nearly uniformly convex (NUC) if for all $\epsilon > 0$, there exists $\delta > 0$ if $x_n \in B_X$, $\text{sep}(x_n) \geq \epsilon$, then $\text{co}(x_n) \cap (1 - \delta)B_X \neq \emptyset$.

It is easy to see that (UKK) $\Rightarrow$ (KK) and (UC) $\Rightarrow$ (NUC). Huff [3] showed that a space is (NUC) if and only if it is (UKK) and reflexive. We have the following implication.

$$(\text{UC}) \Rightarrow (\text{NUC}) \Rightarrow (\text{UKK}) \Rightarrow (\text{KK})$$

For any subset C , we denote by $\alpha(C)$ its Kuratowski measure of non-compactness, i.e., the infimum of such $\epsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ϵ .

For any $x \notin B_X$, the drop determined by x is the set

$$D(x, B_X) = \text{co}(\{x\} \cup B_X)$$

Rolewicz [8] has defined property (β) . A Banach space X is said to have property (β) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\alpha(D(x, B_X) \setminus B_X) < \epsilon$$

whenever $1 < \|x\| < 1 + \delta$.

The following result is found in [5].

A Banach space X has property (β) if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for each element $x \in B_X$ and each sequence $(x_n) \in B_X$ with $\text{sep}(x_n) \geq \epsilon$, there is $k \in \mathbb{N}$ such that

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

By this result we can easily prove that property (β) implies (NUC). We have finally

$$(\text{UC}) \Rightarrow \text{property } (\beta) \Rightarrow (\text{NUC}) \Rightarrow (\text{UKK}) \Rightarrow (\text{KK})$$

A Banach space is said to have the Banach-Saks property if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. S. Kakutani [4] showed that Uniform convexity implies the Banach-Saks property. And T. Nishiura and D. Waterman [7] proved that the Banach-Saks property implies reflexivity in Banach spaces

A Banach space X is called (r, δ) -convex if for any r elements $x_1, x_2, \dots, x_r$ of X with $\|x_i\| \leq 1$ there is an alternate signs sequence $\epsilon = (\epsilon_i)_{i=1}^r$ of ± 1 such that $\|\epsilon_1 x_1 + \dots + \epsilon_r x_r\| \leq r(1 - \delta)$. A Banach space X which is (r, δ) -convex for some r and some $\epsilon > 0$ is called B-convex.

To end with this introduction, let us mention the following lemma.

LEMMA 1 [6]. *Let $(Y, \|\cdot\|)$ be a Banach space with basis $(e_i : i \in I)$ (unconditional if I is noncountable) and such that, for every finite subset J of I ,*

$$\text{if } 0 \leq |\alpha_j| \leq \beta_j, j \in J, \text{ then } \left\| \sum_{j \in J} \alpha_j e_j \right\| \leq \left\| \sum_{j \in J} \beta_j e_j \right\|.$$

Let $(X_i, i \in I)$ be a family of finite dimensional Banach space. Let

$$Z := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_{i \in I} \|x_i\| e_i \in Y \right\}$$

equipped with the norm $\|(x_i)_{i \in I}\| = \left\| \sum_{i \in I} \|x_i\| e_i \right\|$. Then, if $(Y, \|\cdot\|)$ has property (β) , $(Z, \|\cdot\|)$ has property (β) , too.

2. B-convexity and property (β)

We start with the following results.

THEOREM 2 [2]. *If a Banach space X has property (β) , then both X and X^* have the Banach-Saks property.*

THEOREM 3 [1]. *If a Banach space X is reflexive and B -convex, then both X and X^* have the Banach-Saks property.*

By Theorem 2 and Theorem 3, it is natural to consider the relationship between the property (β) in Banach spaces and B -convexity in reflexive Banach spaces. We have another results concerning the properties.

PROPOSITION 4. *If a Banach space X is k -uniformly convex, there exists $\delta > 0$ such that X is (k, δ) -convex.*

Proof. Suppose that X is k -uniformly convex. Then for $\epsilon = 1$, there exists $\delta(1) > 0$ such that if $\|x_i - x_j\| \leq 1$, $i \neq j$, $1 \leq i, j \leq k$ and $x_i \in B_X$, then

$$\left\| \frac{1}{k} \sum_{i=1}^k x_i \right\| \leq 1 - \delta(1)$$

Let $\delta = \inf \left\{ \frac{1}{k}, \delta(1) \right\}$. It suffices to show that for $x_i \in B_X$ and $i = 1, 2, \dots, k$,

$$\inf_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^k \epsilon_i x_i \right\| \leq k(1 - \delta).$$

Suppose that

$$\left\| \sum_{i=1}^k \epsilon_i x_i \right\| > k(1 - \delta),$$

for all signs $\epsilon_i = \pm 1$.

For $1 \leq i < j \leq k$,

$$\begin{aligned} \|x_i - x_j\| &\geq \|x_1 + \dots + x_i + \dots + x_{j-1} - x_j + \dots + x_k\| - \sum_{l \neq i, j} \|x_l\| \\ &> k(1 - \delta) - (k - 2) = 2 - k\delta \geq 1 \end{aligned}$$

Then

$$\left\| \frac{1}{k} \sum_{i=1}^k x_i \right\| < 1 - \delta(1) \leq 1 - \delta.$$

We get the contradiction. $\square$

PROPOSITION 5. *If X is k -uniformly convex then it has property (β) .*

Proof. Suppose that X is k -uniformly convex. Let $\epsilon > 0$. Then there exists $\delta \left(\frac{\epsilon}{3(k-1)} \right) > 0$ such that if $z_i \in B_X$ and $\|z_i - z_j\| \geq \frac{\epsilon}{3(k-1)}$ $i \neq j$ and $1 \leq i, j \leq k$, then $\left\| \frac{1}{k} \sum_{i=1}^k z_i \right\| \leq 1 - \delta \left(\frac{\epsilon}{3(k-1)} \right)$.

We show that there exists $\delta > 0$ such that for $x, x_n \in B_X$ with $\text{sep}(x_n) \geq \epsilon$, there exists $m \in \mathbb{N}$ such that $\left\| \frac{x+x_m}{2} \right\| \leq 1 - \delta$.

Take $\delta = \delta \left(\frac{\epsilon}{3(k-1)} \right)$. Let $x \in B_X$ and $x_n \in B_X$ with $\text{sep}(x_n) \geq \epsilon$. Then there exists $m \in \mathbb{N}$ such that $\|x - x_m\| \geq \frac{\epsilon}{3}$. (Indeed, if for all $n \in \mathbb{N}$ $\|x - x_n\| < \frac{\epsilon}{3}$, then $\|x_i - x_j\| \leq \|x_i - x\| + \|x_j - x\| < \frac{2\epsilon}{3}$. This contradicts that $\text{sep}(x_n) \geq \epsilon$.)

Consider $y_1 = x_m$, $y_2 = \frac{1}{k-1}\{x + (k-2)x_m\}$, $y_3 = \frac{1}{k-1}\{2x + (k-3)x_m\}, \dots$, $y_{k-1} = \frac{1}{k-1}\{(k-2)x + x_m\}$, $y_k = x$. Since for $1 \leq i < j \leq k$

$$\begin{aligned} \|y_i - y_j\| &= \frac{1}{k-1} \|\{(i-1)x + (k-i)x_m\} - \{(j-1)x + (k-j)x_m\}\| \\ &= \frac{1}{k-1} \|(i-j)x + (j-i)x_m\| = \frac{(j-i)}{(k-1)} \|x_m - x\| \\ &\geq \frac{1}{k-1} \|x_m - x\| \geq \frac{\epsilon}{3(k-1)} \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^k y_i &= \frac{1}{k-1} \sum_{i=1}^k \{(i-1)x + (k-i)x_m\} \\ &= \frac{1}{(k-1)} \left\{ x \frac{(k-1)k}{2} + x_m \left(k^2 - \frac{k(k+1)}{2} \right) \right\} \\ &= \frac{k}{2}x + \frac{k}{2}x_m = \frac{k(x+x_m)}{2}, \end{aligned}$$

$$\left\| \frac{x+x_m}{2} \right\| = \left\| \frac{1}{k} \sum_{i=1}^k y_i \right\| \leq 1 - \delta \left(\frac{\epsilon}{3(k-1)} \right) = 1 - \delta.$$

This completes our proof. $\square$

By Proposition 4 and 5, k -uniformly convexity implies property (β) and B-convexity.

We can get a simple example which is B-convex and does not have property (β) , by renorming $(l_2, \|\cdot\|_2)$.

EXAMPLE 6. Define a norm $\|\cdot\|$ in l_2 by

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| = \max \left\{ |a_1|, \left\| \sum_{n=2}^{\infty} a_n e_n \right\|_2 \right\},$$

where (e_n) is a usual unit vector basis of l_2 . Then $(l_2, \|\cdot\|)$ and $(l_2, \|\cdot\|_2)$ are isomorphic, since $\|\sum_{n=1}^{\infty} a_n e_n\| \leq \|\sum_{n=1}^{\infty} a_n e_n\|_2 \leq 2 \|\sum_{n=1}^{\infty} a_n e_n\|$. Since $(l_2, \|\cdot\|_2)$ is uniformly convex, $(l_2, \|\cdot\|)$ is superreflexive. Since superreflexivity implies B-convexity and reflexivity, $(l_2, \|\cdot\|)$ is reflexive and B-convex.

We show that $(l_2, \|\cdot\|)$ has no property (β) . It suffices to show that X is not (KK), since $(\text{UC}) \Rightarrow \text{property } (\beta) \Rightarrow (\text{NUC}) \Rightarrow (\text{UKK}) \Rightarrow (\text{KK})$.

Consider $x_n = e_1 + e_{n+1}$. Then $x_n \rightarrow e_1$ weakly (Indeed, for $x^* = \sum_{n=1}^{\infty} \alpha_n e_n \in (l_2, \|\cdot\|)^* = (l_2, \|\cdot\|_2)^*$, $x^*(x_n) = \alpha_1 + \alpha_{n+1} \rightarrow x^*(e_1) = \alpha_1$) but $x_n \not\rightarrow e_1$ in norm, since $\|x_n - e_1\| = \|e_{n+1}\| = 1$. This implies that $(l_2, \|\cdot\|)$ is not (KK).

It is well known fact that B-convexity is isomorphic invariant. We can see that property (β) is not isomorphic invariant by Example 6. Thus it is reasonable to consider the following definition.

DEFINITION 7. A Banach space $(X, \|\cdot\|)$ is called a (β) -space if there is a norm $\|\cdot\|_1$ equivalent to $\|\cdot\|$ such that $(X, \|\cdot\|_1)$ satisfies property (β) .

Every superreflexive space is a (β) -space [8] but the converse implication does not hold [6]. It is well known fact that superreflexive space is B-convex and reflexive. Since (β) -space is reflexive, it is an apparent question whether (β) -space is B-convex.

THEOREM 8. *There exists a (β) -space which cannot be equivalently renormed to be B-convex.*

Proof. Since B-convexity is norm isomorphic invariant, it is enough to find non-B-convex space which satisfies (β) -property. Let

$$Z = \left\{ (x_i) \in \prod_{i=1}^{\infty} \mathbb{R}^i : \sum_{i=1}^{\infty} \|x_i\|_{\infty} e_i \in l_2, x_i \in \mathbb{R}^i \right\}$$

equipped with the norm $\|(x_i)\| = \|\sum_{i=1}^{\infty} \|x_i\|_{\infty} e_i\|_2$ where (e_n) is usual unit vector basis of l_2 . Then Z has property (β) by Lemma 1. We prove that Z is not B-convex. It suffices to show that for all $n \in \mathbb{N}$ there exist $x^{(1)}, x^{(2)}, \dots, x^{(n)} \in Z$ such that $\|x^{(k)}\| = 1$, $k = 1, 2, \dots, n$ and $\|\sum_{k=1}^n \epsilon_k x^{(k)}\| = n$, for all $\epsilon_k = \pm 1$.

Define $x^{(k)} = (x_i^{(k)}) \in Z$ with

$$x_i^{(k)} = \begin{cases} \sum_{j=1}^{2^{n-1}} \frac{1}{\sqrt{n}} \epsilon_j^{(k)} e_j^{(i)}, & 2^{n-1} \leq i \leq 2^{n-1} + n - 1 \\ 0 \in \mathbb{R}^i, & \text{elsewhere,} \end{cases}$$

if $\epsilon_j^{(1)} = 1$, $1 \leq j \leq 2^{n-1}$, for $k \geq 2$

$$\epsilon_j^{(k)} = \begin{cases} 1, & (2l-2)2^{n-k} < j \leq (2l-1)2^{n-k} \\ -1, & (2l-1)2^{n-k} < j \leq 2l \cdot 2^{n-k}, \quad 1 \leq l \leq 2^{k-2} \end{cases}$$

and $(e_j^{(i)})_{j=1}^{j=i}$ is usual unit vector basis of $\mathbb{R}^i$.

Then for $k = 1, 2, \dots, n$,

$$\|x^{(k)}\| = \|(x_i^{(k)})\| = \left(\sum_{i=1}^{\infty} \|x_i^{(k)}\|_{\infty}^2 \right)^{\frac{1}{2}} = \left(\sum_{i=2^{n-1}}^{2^{n-1}+n-1} \left(\frac{1}{\sqrt{n}} \right)^2 \right)^{\frac{1}{2}} = 1$$

and for $\epsilon_k = \pm 1$, $k = 1, 2, \dots, n$,

$$\begin{aligned} \|\epsilon_1 x^{(1)} + \dots + \epsilon_n x^{(n)}\| &= \left\| \sum_{i=1}^{\infty} \|\epsilon_1 x_i^{(1)} + \dots + \epsilon_n x_i^{(n)}\|_{\infty} e_i \right\|_2 \\ &= \left\| \sum_{i=2^{n-1}}^{2^{n-1}+n-1} \|\epsilon_1 x_i^{(1)} + \dots + \epsilon_n x_i^{(n)}\|_{\infty} e_i \right\|_2 \\ &= \left\| \sum_{i=2^{n-1}}^{2^{n-1}+n-1} \frac{n}{\sqrt{n}} e_i \right\|_2 = n. \end{aligned}$$

This implies that Z is not (n, δ) -convex for all $n \in \mathbb{N}$, $\delta > 0$. This completes our proof. $\square$

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