ON C-INTEGRAL OF BANACH-VALUED FUNCTIONS

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ABSTRACT. In this paper, we define and study the C-integral and the strong C-integral of functions mapping an interval \([a,b]\) into a Banach space \(X\). We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional. We also consider the property of primitives corresponding to Banach-valued functions with respect to the C-integral and the strong C-integral.

1. Introduction

It is well known that the Henstock integral is a kind of non-absolute integral and includes the Riemann, improper Riemann, Newton, and Lebesgue integrals. From the following function

\[
F(x) = \begin{cases} 
\frac{x \sin \frac{1}{x}}{x} & \text{if } 0 < x \leq 1, \\
0 & \text{if } x = 0,
\end{cases}
\]

we know that it is a primitive for the Henstock integral, but it is neither a Lebesgue primitive, neither a differentiable function, nor a sum of Lebesgue primitive and a differentiable function. It is natural to ask that is there a minimal integral includes the Lebesgue integral and the derivative?

In 1996 B. Bongiorno provided a constructive minimal integration process of Riemann type, called C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable function.

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B. Bongiorno and L. Di Piazza [1, 2, 6] discussed some properties of the C-integral of real-valued functions.

In this paper, we define and study the C-integral and the strong C-integral of functions mapping an interval \([a, b]\) into a Banach space \(X\). We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional. If the function has the property \(S^* C\) then it is strongly C-integrable, but the converse is not true. We also consider the property of primitives corresponding to Banach-valued functions with respect to the C-integral and the strong C-integral.

2. Definitions and basic properties

Throughout this paper, \(I_0 = [a, b]\) is a compact interval in \(R\). \(X\) will denote a real Banach space with norm \(\|\cdot\|\) and its dual \(X^*\). A partition \(D\) is a finite collection of interval-point pairs \(\{(I_i, \xi_i)\}_{i=1}^n\), where \(\{I_i\}_{i=1}^n\) are non-overlapping subintervals of \(I_0\), \(\delta(\xi)\) is a positive function on \(I_0\), i.e. \(\delta(\xi) : I_0 \to R^+\). We say that \(D = \{(I_i, \xi_i)\}_{i=1}^n\) is

1. a partial partition of \(I_0\) if \(\bigcup_{i=1}^n I_i \subset I_0\),
2. a partition of \(I_0\) if \(\bigcup_{i=1}^n I_i = I_0\),
3. \(\delta\)-fine McShane partition of \(I_0\) if \(I_i \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))\) and \(\xi_i \in I_0\) for all \(i = 1, 2, \cdots, n\),
4. \(\delta\)-fine C-partition of \(I_0\) if it is a \(\delta\)-fine McShane partition of \(I_0\) and satisfying the condition

\[
\sum_{i=1}^n \text{dist}(\xi_i, I_i) < \frac{1}{\varepsilon}
\]

for the given \(\varepsilon\), here \(\text{dist}(\xi_i, I_i) = \inf\{|t_i - \xi_i| : t_i \in I_i\}\),

(5) \(\delta\)-fine partition of \(I_0\) if \(\xi_i \in I_i \subset B(\xi_i, \delta(\xi_i))\) for all \(i = 1, 2, \cdots, n\).

Given a \(\delta\)-fine C-partition (McShane partition) \(D = \{(I_i, \xi_i)\}_{i=1}^n\) we write

\[
S(f, D) = \sum_{i=1}^n f(\xi_i)|I_i|
\]

for integral sums over \(D\), whenever \(f : I_0 \to X\).

**Definition 2.1.** A function \(f : I_0 \to X\) is C-integrable if there exists a vector \(A \in X\) such that for each \(\varepsilon > 0\) there is a positive function
δ(ξ) : I₀ → R⁺ such that

\[ \|S(f, D) - A\| < \epsilon \]

for each δ - fine C-partition \( D = \{(I_i, \xi_i)\}_{i=1}^n \) of \( I₀ \). \( A \) is called the C-integral of \( f \) on \( I₀ \), and we write \( A = \int_{I₀} f \) or \( A = (C) \int_{I₀} f \).

The function \( f \) is C-integrable on the set \( E \subset I₀ \) if the function \( fχ_E \) is C-integrable on \( I₀ \). We write \( \int_E f = \int_{I₀} fχ_E \).

We can easily get the following two theorems.

**Theorem 2.1.** A function \( f : I₀ \rightarrow X \) is C-integrable if and only if for each \( \epsilon > 0 \) there is a positive function \( \delta(\xi) : I₀ \rightarrow R⁺ \) such that

\[ \|S(f, D₁) - S(f, D₂)\| < \epsilon \]

for arbitrary \( \delta \)-fine C-partition \( D₁ \) and \( D₂ \) of \( I₀ \).

**Theorem 2.2.** Let \( f : I₀ \rightarrow X \) and \( g : I₀ \rightarrow X \).

1. if \( f \) is C-integrable on \( I₀ \), then \( f \) is C-integrable on every subinterval of \( I₀ \).
2. if \( f \) is C-integrable on each of the intervals \( I₁ \) and \( I₂ \), where \( I_i \) are non-overlapping and \( I₁ \cup I₂ = I₀ \), then \( f \) is C-integrable on \( I₀ \) and \( \int_{I₁} f + \int_{I₂} f = \int_{I₀} f \).
3. if \( f \) and \( g \) are C-integrable on \( I₀ \) and if \( \alpha \) and \( \beta \) are real numbers, then \( \alpha f + \beta g \) is C-integrable on \( I₀ \) and \( \int_{I₀} (\alpha f + \beta g) = \alpha \int_{I₀} f + \beta \int_{I₀} g \).

**Lemma 2.3.** (Saks-Henstock) Let \( f : I₀ \rightarrow X \) is C-integrable on \( I₀ \). Then for each \( \epsilon \) there is a positive function \( \delta(\xi) : I₀ \rightarrow R⁺ \) such that

\[ \|S(f, D) - \int_{I₀} f\| < \epsilon \]

for each \( \delta \)-fine C-partition \( D = \{(I_i, \xi_i)\} \) of \( I₀ \). Particularly, if \( D' = \{(I_i, \xi_i)\}_{i=1}^m \) is an arbitrary \( \delta \)-fine partial C-partition of \( I₀ \), we have

\[ \|S(f, D') - \sum_{i=1}^m \int_{I_i} f(\xi_i)\| \leq \epsilon. \]

**Proof.** The proof is similar to the case for Banach-valued Henstock integrable functions and the reader is referred to [7, Lemma 3.4.1.] for details. □
Theorem 2.4. Let \( f : I_0 \to X \), if \( f = \theta \) almost everywhere on \( I_0 \), then \( f \) is C-integrable on \( I_0 \) and \( \int_{I_0} f = \theta \).

Proof. Let \( \epsilon \) be given. Assume \( E = \{ \xi \in I_0 : f(\xi) \neq \theta \} \) and \( E = \bigcup E_n \subset I_0 \) where \( E_n = \{ \xi \in I_0 : n-1 \leq \|f(\xi)\| < n \} \). Obviously, \( \mu(E) = 0 \) and therefore \( \mu(E_n) = 0 \), then there are open sets \( G_n \subset I_0 \) such that \( E_n \subset G_n \) and \( \mu(G_n) < \frac{\epsilon}{n^2} \). We define a positive function \( \delta(\xi) : I_0 \to R^+ \) in such a way that \( \delta(\xi) = 1 \) for \( \xi \in I_0 \setminus E \) and \( B(\xi, \delta(\xi)) \subset G_n \) if \( \xi \in E_n \).

Suppose that \( D = \{(I, \xi)\} \) is a \( \delta \) - fine C-partition of \( I_0 \), then

\[
\| \sum_{(I, \xi)} f(\xi) |I| \| \leq \sum n \cdot \frac{\epsilon}{n^2} \leq \epsilon.
\]

Hence \( f \) is C-integrable on \( I_0 \) and \( \int_{I_0} f = \theta \). \( \square \)

Corollary 2.5. Let \( f : I_0 \to X \) is C-integrable on \( I_0 \). If \( f = g \) almost everywhere on \( I_0 \), then \( g \) is C-integrable on \( I_0 \) and \( \int_{I_0} f = \int_{I_0} g \) almost everywhere on \( I_0 \).

Theorem 2.6. Let \( f : I_0 \to X \) is C-integrable on \( I_0 \).

(1) for each \( x^* \in X^* \), the function \( x^* f \) is C-integrable on \( I_0 \) and \( \int_{I_0} x^* f = x^*(\int_{I_0} f) \).

(2) If \( T : X \to Y \) is a continuous linear operator, then \( Tf \) is C-integrable on \( I_0 \) and \( \int_{I_0} Tf = T(\int_{I_0} f) \).

Proof. (1) Since \( f : I_0 \to X \) is C-integrable on \( I_0 \), for each \( \epsilon > 0 \) there is a positive function \( \delta(\xi) : I_0 \to R^+ \) such that

\[
\| S(f, D) - \int_{I_0} f \| < \frac{\epsilon}{\|x^*\|}
\]

for each \( \delta \) - fine C-partition \( D = \{(I, \xi)\} \) of \( I_0 \). Hence for each \( x^* \in X^* \), we have

\[
\| S(x^* f, D) - x^*(\int_{I_0} f) \| \leq \|x^*\| \cdot \| S(f, D) - \int_{I_0} f \| < \epsilon.
\]

(2) \( T : X \to Y \) is a continuous linear operator, then there exists a number \( M > 0 \) such that \( \|Tx\| \leq M\|x\| \) for each \( x \in X \). Since \( f : I_0 \to X \) is C-integrable on \( I_0 \), for each \( \epsilon > 0 \) there is a positive function \( \delta(\xi) : I_0 \to R^+ \) such that

\[
\| S(f, D) - \int_{I_0} f \| < \frac{\epsilon}{M}
\]
for each $\delta$ - fine C-partition $D = \{(I, \xi)\}$ of $I_0$. Hence we have
$$\|S(Tf, D) - T(\int_{I_0} f)\| \leq M \cdot \|S(f, D) - \int_{I_0} f\| < \epsilon.$$ 

\[\square\]

3. C-integral and strong C-integral

**Definition 3.1.** A function $f : I_0 \rightarrow X$ is strongly C-integrable if there exists an additive function $F : I_0 \rightarrow X$ such that for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \rightarrow R^+$ such that
$$\sum_{i=1}^n \|f(\xi_i)I_i - F(I_i)\| < \epsilon$$
for each $\delta$ - fine C-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $I_0$. $F(I_0)$ is called the strong C-integral of $f$ on $I_0$, and we write $F(I_0) = \int_{I_0} f$.

**Theorem 3.1.** Let $f : I_0 \rightarrow X$ is strongly C-integrable on $I_0$, then $f$ is C-integrable on $I_0$.

**Proof.** It follows from the definitions of the strong C-integral and C-integral that if $f$ is strongly C-integrable on $I_0$, then $f$ is C-integrable on $I_0$. 

**Definition 3.2.** A function $f : I_0 \rightarrow X$ is strongly Henstock (McShane) integrable if there exists an additive function $F : I_0 \rightarrow X$ such that for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \rightarrow R^+$ such that
$$\sum_{i=1}^n \|f(\xi_i)I_i - F(I_i)\| < \epsilon$$
for each $\delta$ - fine (McShane) partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $I_0$. $F(I_0)$ is called the strong Henstock (McShane) integral of $f$ on $I_0$, and we write $F(I_0) = \int_{I_0} f$.

By the definitions of strong Henstock, strong C-integral and strong McShane integral and the fact that each $\delta$ - fine partition is also $\delta$ - fine C-partition and therefore is also $\delta$ - fine McShane partition, we get immediately the following Theorem.
THEOREM 3.2. Let the function $f : I_0 \to X$.
(a) if $f$ is strongly McShane integrable then $f$ is strongly C-integrable.
(b) if $f$ is strongly C-integrable then $f$ is strongly Henstock integrable.

THEOREM 3.3. If a Banach space $X$ is finite dimensional then a function $f : I_0 \to X$ is C-integrable on $I_0$ if and only if $f$ is strongly C-integrable on $I_0$.

Proof. We only prove if $X$ is finite dimensional and $f : I_0 \to X$ is C-integrable on $I_0$ then $f$ is strongly C-integrable on $I_0$.

Let $\epsilon$ be given. It follows from the definition of the C-integral that there is a positive function $\delta : I_0 \to \mathbb{R}^+$ such that
\[
\| \sum f(\xi) I - F(I) \| < \epsilon
\]
for each $\delta$-fine C-partition $D = \{(I, \xi)\}$ of $I_0$. Let $\{e_1, e_2, \cdots, e_n\}$ be a base of $X$ and $g_i : I_0 \to \mathbb{R}$ ($i = 1, 2, \cdots, n$). By the Hahn-Banach Theorem, for each $e_i$ there is $x_i^* \in X^*$ such that
\[
(2) \quad x_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]
for $i, j = 1, 2, \cdots, n$ and therefore $x_i^*(f) = \sum_{j=1}^n g_j x_i^*(e_j) = g_i$. Since $g_i : I_0 \to \mathbb{R}$ is C-integrable on $I_0$ from Theorem 2.6, there is a positive function $\delta_i(\xi) : I_0 \to \mathbb{R}^+$ such that
\[
|S(g_i, D_i) - \sum \int_I g_i| < \epsilon
\]
for each $\delta_i$-fine C-partition $D_i = \{(I, \xi)\}$ of $I_0$. By an easy adaptation of Saks-Henstock Lemma we have
\[
\sum |g_i(\xi)| I - \int_I g_i| < 2\epsilon.
\]
We also have
\[
F(I) = \int_I f = \int_I \sum_{i=1}^n g_i e_i = \sum_{i=1}^n \int_I g_i e_i = \sum_{i=1}^n e_i G_i(I)
\]
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where \( G_i(I) = \int_I g_i \). Let \( \delta(\xi) < \delta_i(\xi) \) for \( i = 1, 2, \cdots, n \) and consequently

\[
\sum \| f(\xi)|I| - F(I) \| = \sum_{i=1}^n \| \sum_{i=1}^n g_i(\xi)e_i|I| - \sum_{i=1}^n e_i G_i(I) \|
\leq \sum_{i=1}^n \| e_i \| \sum_{i=1}^n |g_i(\xi)| |I| - G_i(I)|
\leq \varepsilon \sum_{i=1}^n \| e_i \|
\]

for each \( \delta \)-fine C-partition \( D = \{(I, \xi)\} \) of \( I_0 \). Hence \( f \) is strongly C-integrable on \( I_0 \).

\[ \Box \]

**Theorem 3.4.** If \( X \) is a Banach space then the C-integral is equivalent to the strong C-integral on \( I_0 \) if and only if \( X \) is finite dimensional.

**Proof.** The proof of necessity is similar to the case for Henstock (McShane) integral, see, for example, [8, Theorem 3]. In [8], if \( X \) is infinite dimensional, there exist \( x_r^1, x_r^2, \cdots, x_r^2r \in X \) such that

\[
\| x_r^i \| = \frac{1}{2^r}
\]

for each \( r, 1 \leq i \leq 2r \) and we also have

\[
\| \sum_{i=1}^{2r} \theta_r^i x_r^i \|^2 \leq \frac{3}{2^r}
\]

for every \( \theta_r^i \) with \( |\theta_r^i| \leq 1, 1 \leq i \leq 2^r \). V.A. Svortsov and A.P. Solodov defined a function \( f : [0, 1] \rightarrow X \) in the following way

\[
f(t) = \begin{cases} 0 & \text{if } t \in C, \text{ or } t = d_r^i, r \geq 0, 1 \leq i \leq 2^r, \\ 2 \cdot 3^r x_r^i & \text{if } t \in (a_r^i, d_r^i), r \geq 0, 1 \leq i \leq 2^r, \\ -2 \cdot 3^r x_r^i & \text{if } t \in (d_r^i, b_r^i), r \geq 0, 1 \leq i \leq 2^r. 
\end{cases}
\]

(3) \( f(t) = \)

Here \( C \) be the cantor ternary set, \( (a_r^i, b_r^i), r \geq 0, 1 \leq i \leq 2^r \) being the intervals of rank \( r \) contiguous to \( C \) with middle points \( d_r^i \) and satisfying \( b_r^i - a_r^i = 3^{-(r+1)} \).

The function is McShane integrable but is not strongly Henstock integrable, then we have \( f \) is C-integrable and is not strongly C-integrable.
In other words, if the C-integral is equivalent to the strong C-integral then $X$ is finite dimensional.

**Definition 3.3.** A function $F : I_0 \to X$ is differentiable at $\xi \in I_0$ if there is a $f(\xi) \in X$ such that

$$
\lim_{\delta \to 0} \frac{\| F(\xi + \delta) - F(\xi) \| - f(\xi) \|}{\delta} = 0.
$$

We denote $f(\xi) = F'(\xi)$ the derivative of $F$ at $\xi$.

**Theorem 3.5.** If a function $F : I_0 \to X$ is differentiable on $I_0$ with $f(\xi) = F'(\xi)$ for each $\xi \in I_0$, then $f : I_0 \to X$ is strongly C-integrable.

**Proof.** Let $\epsilon$ be given. By the definition of derivative, for each $\xi \in I_0$ there is a positive function $\delta(\xi) : I_0 \to \mathbb{R}^+$ such that

$$
\| \frac{F(\zeta) - F(\xi)}{\zeta - \xi} - f(\xi) \| < \frac{\epsilon^2}{2(1 + \epsilon|I_0|)}
$$

for all $\zeta \in I_0$ with $|\zeta - \xi| < \delta(\xi)$. Assume $D = \{(I_i, \xi_i)\}_{i=1}^m$ is an arbitrary $\delta$-fine C-partition of $I_0$, we have

$$
\sum_{i=1}^n \| f(\xi_i)|I_i| - F(I_i) \| < \frac{\epsilon^2}{1 + \epsilon|I_0|} \sum_{i=1}^n (\text{dist}(\xi_i, I_i) + |I_i|)
$$

$$
< \frac{\epsilon^2}{1 + \epsilon|I_0|} \left( \frac{1}{\epsilon} + |I_0| \right) < \epsilon.
$$

Hence $f : I_0 \to X$ is strongly C-integrable on $I_0$.  

**Definition 3.4.** A function $f : I_0 \to X$ has the property $S^*C$ if for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$
\sum_{i=1}^m \sum_{j=1}^n \| f(\xi_i) - f(\xi_j) \| : |I_i \cap I_j| < \epsilon
$$

for arbitrary $\delta$-fine C-partitions $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of $I_0$.

**Theorem 3.6.** If a function $f : I_0 \to X$ has the property $S^*C$ then $f$ is strongly C-integrable on $I_0$.

**Proof.** We will prove the Theorem in two steps.
Step 1. Let $\epsilon$ be given. Assume $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ are arbitrary $\delta$-fine C-partition of $I_0$ then we have

$$\left\| \sum_{i=1}^m f(\xi_i)|I_i| - \sum_{j=1}^n f(\zeta_j)|L_j| \right\|$$

$$= \left\| \sum_{j=1}^n \sum_{i=1}^m f(\xi_i)|I_i \cap L_j| - \sum_{i=1}^m \sum_{j=1}^n f(\zeta_j)|I_i \cap L_j| \right\|$$

$$= \left\| \sum_{i=1}^m \sum_{j=1}^n (f(\xi_i) - f(\zeta_j))|I_i \cap L_j| \right\|$$

$$\leq \sum_{i=1}^m \sum_{j=1}^n \|f(\xi_i) - f(\zeta_j)\| \cdot |I_i \cap L_j| < \epsilon.$$  

By Theorem 2.1 we have $f$ is C-integrable on $I_0$.

Step 2. By Definition 3.4, for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \rightarrow \mathbb{R}^+$ such that

$$\sum_{i=1}^m \sum_{j=1}^n \|f(\xi_i) - f(\zeta_j)\| \cdot |I_i \cap L_j| < \epsilon$$

for arbitrary $\delta$-fine C-partition $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of $I_0$. $f$ is C-integrable on $I_0$ and therefore C-integrable on $I_i$ for $i = 1, 2, \cdots, m$. Hence for given $\epsilon > 0$ there is a positive function $\delta_i(\xi) : I_0 \rightarrow \mathbb{R}^+$ such that $\delta_i(\xi) \leq \delta(\xi)$ and such that for any $\delta_i$-fine C-partition $D_i = \{(L^i_j, \zeta^i_j)\}_{j=1}^{n^i}$ for $i = 1, 2, \cdots, m$ of $I_i$, then

$$\left\| \sum_{j=1}^{n^i} f(\zeta^i_j)|L^i_j| - \int_{I_i} f \right\| = \left\| \sum_{j=1}^{n^i} [f(\zeta^i_j)|L^i_j| - \int_{L^i_j} f] \right\| < \frac{\epsilon}{2m}.$$
Assume $D = \bigcup_{i=1}^{m} D_i$, obviously it is a $\delta$-fine C-partition of $I_0$. Hence we have

$$
\sum_{i=1}^{m} \|f(\xi_i)I_i - F(I_i)\|
= \sum_{i=1}^{m} \| \sum_{j=1}^{n_i} f(\xi_i)I_i \bigcap L_{ij}^i - \sum_{j=1}^{n_i} F(I_i \bigcap L_{ij}^i)\|
= \sum_{i=1}^{m} \| \sum_{j=1}^{n_i} (f(\xi_i) - f(\zeta_{ij}^i))I_i \bigcap L_{ij}^i + \sum_{j=1}^{n_i} f(\zeta_{ij}^i)I_i \bigcap L_{ij}^i - F(I_i \bigcap L_{ij}^i)\|
\leq \sum_{i=1}^{m} \sum_{j=1}^{n_i} \|f(\xi_i) - f(\zeta_{ij}^i)\| \cdot |I_i \bigcap L_{ij}^i|
+ \sum_{i=1}^{m} \| \sum_{j=1}^{n_i} [f(\zeta_{ij}^i)I_i \bigcap L_{ij}^i - F(I_i \bigcap L_{ij}^i)]\|
\leq \frac{\epsilon}{2} + \sum_{i=1}^{m} \frac{\epsilon}{2m} = \epsilon.
$$

By Definition 3.1, $f$ is strongly C-integrable on $I_0$.

**Remark 3.1.** The converse of Theorem 3.6 is not true, in other words, if $f$ is strongly C-integrable on $I_0$, it does not necessarily have the property $S^C$.

**Proof.** Assume a real-valued function $f$ by

$$
f(x) = \begin{cases} 
2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\
0 & \text{if } x = 0.
\end{cases}
$$

It is easy to know that the primitive of $f$ is

$$
F(x) = \begin{cases} 
x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\
0 & \text{if } x = 0.
\end{cases}
$$

$F(x)$ is differentiable everywhere and $F'(x) = f(x)$ everywhere on $[0, 1]$, then $f(x)$ is strongly C-integrable on $[0, 1]$ from Theorem 3.5.
Assume $f$ has the property $S^*C$, then for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [0, 1] \to R^+$ such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |f(\xi_i) - f(\xi_j)| \cdot |I_i \cap L_j| < \varepsilon$$

for arbitrary $\delta$-fine C-partitions $D_1 = \{(I_i, \xi_i)\}_{i=1}^{m}$ and $D_2 = \{(L_j, \xi_j)\}_{j=1}^{n}$ of $[0, 1]$. Consequently we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} ||f(\xi)| - |f(\xi_j)|| \cdot |I_i \cap L_j| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |f(\xi_i) - f(\xi_j)| \cdot |I_i \cap L_j| < \varepsilon$$

and therefore $|f(x)|$ has the property $S^*C$, then $|f(x)|$ is C-integrable on $[0, 1]$ from Theorem 3.6 and Theorem 3.1. We claim that $f(x)$ is Lebesgue (McShane) integrable, the proof is too easy and will be omitted. But in fact $F(x)$ is not absolutely continuous on $[0, 1]$ and therefore $f(x)$ is not Lebesgue integrable on $[0, 1]$, then $f(x)$ does not have the property $S^*C$. \hfill \Box

4. Primitive of the C-integral and strong C-integral

**Definition 4.1.** Let $F : I_0 \to X$ and let $E$ be a subset of $I_0$.

(a) $F$ is said to be $AC_{c}$ on $E$ if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\sum_i F(I_i) < \varepsilon$ for each $\delta$-fine partial C-partition $D = \{(I_i, \xi_i)\}$ of $I_0$ satisfying $\xi_i \in E$ for each $i$ and $\sum_i |I_i| < \eta$.

(b) $F$ is said to be $AC_{c}$ on $E$ if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\sum_i F(I_i) < \varepsilon$ for each $\delta$-fine partial C-partition $D = \{(I_i, \xi_i)\}$ of $I_0$ satisfying $\xi_i \in E$ for each $i$ and $\sum_i |I_i| < \eta$.

(c) $F$ is said to be $AC_{c}^*$ on $E$ if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\sum_i \omega(F, I_i) < \varepsilon$ for each $\delta$-fine partial C-partition $D = \{(I_i, \xi_i)\}$ of $I_0$ satisfying $\xi_i \in E$ for each $i$ and $\sum_i |I_i| < \eta$.

(d) $F$ is said to be $AC_{c}^*$ on $E$ if $F$ is continuous on $E$ and $E$ can be expressed as a countable union of sets on each of which $F$ is $AC_{c}^*$.

**Theorem 4.1.** If a function $f : I_0 \to X$ is C-integrable on $I_0$ with the primitive $F : I_0 \to X$, then $F$ is $AC_{c}$ on $I_0$. 

Proof. Since $f$ is C-integrable on $I_0$, by the Saks-Henstock Lemma, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\| \sum_{i=1}^{m} [f(\xi_i)|I_i| - F(I_i)] \| \leq \varepsilon$$

for each $\delta$-fine partial C-partition $D = \{(I_i, \xi_i)\}_{i=1}^{m}$ of $I_0$.

Let $E_n = \{\xi \in I_0 : n - 1 \leq \|f(\xi)\| < n\}$ for $n = 1, 2, \ldots$. Obviously, the union of $E_n$ is the whole interval $I_0$. Assume that $\xi_i \in E_n$ for each $i$ and $\sum_i |I_i| < \frac{\varepsilon}{n}$. Then we have

$$\| \sum_{i=1}^{m} F(I_i) \| \leq \| \sum_{i=1}^{m} [F(I_i) - f(\xi_i)|I_i|] \| + \| \sum_{i=1}^{m} f(\xi_i) \cdot |I_i| \|$$

$$\leq \varepsilon + n \cdot \sum_{i=1}^{m} |I_i| < 2\varepsilon.$$

This shows that $F$ is $^*AC_c$ on $E_n$. Hence $F$ is $^*ACG_c$ on $I_0$.

\[\square\]

**Theorem 4.2.** If a function $f : I_0 \to X$ is strongly C-integrable on $I_0$ with the primitive $F : I_0 \to X$, then $F$ is $ACG_c^*$ on $I_0$.

**Proof.** Since $f$ is strongly C-integrable on $I_0$, by the Saks-Henstock Lemma, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\sum_{i=1}^{m} \| f(\xi_i)|I_i| - F(I_i) \| \leq \varepsilon$$

for each $\delta$-fine partial C-partition $D = \{(I_i, \xi_i)\}_{i=1}^{m}$ of $I_0$.

It is easy to get that the primitive $F$ is continuous on $I_0$. Thus, for each $i = 1, 2, \ldots, m$ there exists $I_i' \subset I_i$ such that the oscillation $\omega(F, I_i)$ of $F$ on $I_i$ equals $\| F(I_i') \|$, i.e., $\omega(F, I_i) = \| F(I_i') \|$. Consequently, we can get a new $\delta$-fine partial C-partition $D_0 = \{(I_i', \xi_i)\}_{i=1}^{m}$ of $I_0$. Using the Saks-Henstock Lemma again, we have

$$\sum_{i=1}^{m} \| f(\xi_i)|I_i'| - F(I_i') \| \leq \varepsilon.$$
Let \( E_n = \{ \xi \in I_0 : n - 1 \leq \| f(\xi) \| < n \} \) for \( n = 1, 2, \cdots \). Obviously, \( I_0 = \bigcup_n E_n \). Assume that \( \xi_i \in E_n \) for each \( i \) and \( \sum_i |I_i| < \frac{\xi}{n} \). We have

\[
\sum_{i=1}^m \omega(F, I_i) = \sum_{i=1}^m \| F(I_i) \| \\
\leq \sum_{i=1}^m \| F(I_i) - f(\xi) \| I_i \| + \sum_{i=1}^m \| f(\xi) \| \cdot |I_i| \\
\leq \epsilon + n \cdot m |I_i| < 2\epsilon.
\]

Hence \( F \) is \( AC^*_c \) on \( E_n \). Consequently, \( F \) is \( ACG^*_c \) on \( I_0 \).

\[ \square \]

**Theorem 4.3.** If a function \( F : I_0 \to X \) is \( ACG^*_c \) on \( I_0 \) and \( f(\xi) = F'(\xi) \) almost everywhere on \( I_0 \), then \( f : I_0 \to X \) is strongly \( C \)-integrable on \( I_0 \) with the primitive \( F : I_0 \to X \).

**Proof.** There exists a set \( E \subset I_0 \) be of measure zero such that \( f(\xi) = F'(\xi) \) for \( \xi \in I_0 \setminus E \). For \( \xi \in I_0 \setminus E \), by the definition of derivative, for each \( \epsilon > 0 \), there is a positive function \( \delta(\xi) : I_0 \to \mathbb{R}^+ \) such that

\[
\| f(\xi) |I| - F(I) \| < \epsilon |I|
\]

for \( |I| < \delta(\xi) \).

\( F \) is \( ACG^*_c \), then there is a sequence of closed set \( \{ E_i \} \) such that \( I_0 = \bigcup_i E_i \) and \( F \) is \( AC^*_c \) on each \( E_i \). Let \( E_{i,j} = \{ \xi \in E_i \cap Y_j : j - 1 \leq \| f(\xi) \| < j \} \) where \( Y_1 = E_1, Y_i = E_i \setminus (E_1 \cup E_2 \cup \cdots \cup E_{i-1}) \) for \( i = 2, 3, \cdots \). Obviously \( E_{i,j}, j = 1, 2, \cdots \) are pairwise disjoint and \( E = \bigcup E_{i,j} \). \( F \) is also \( AC^*_c \) on \( E_{i,j} \), So there is a \( \eta_{i,j} < \frac{\epsilon}{2^{\delta(i+j)}} \) such that

\[
\sum_k \| F(I_k) \| < \epsilon \cdot 2^{-(i+j)}
\]

for arbitrary \( \delta \) - fine partial \( C \)-partition \( D = \{ (I_k, \xi_k) \} \) satisfying at least one endpoint of \( I_k \) belonging to \( E_{i,j} \) and \( \sum_k |I_k| < \eta_{i,j} \).

There are open intervals \( G_{i,j} \) such that \( E_{i,j} \subset G_{i,j} \) and \( |G_{i,j}| < \eta_{i,j} \). Now for \( \xi \in E_{i,j}, i = 1, 2, \cdots \), put \( \delta(\xi) > 0 \) such that \( B(\xi, \delta(\xi)) \subset G_{i,j} \). Hence we have defined a positive function \( \delta(\xi) \) on \( I_0 \). Splitting the sum
\[
\sum \|f(\xi_i)|I_i| - F(I_i)\|
\]

\[
= \sum_{l, \xi \in I_0 \setminus E} \|f(\xi_i)|I_i| - F(I_i)\| + \sum_{l, \xi \in E} \|f(\xi_i)|I_i| - F(I_i)\|
\]

\[
\leq \sum_{l, \xi \in I_0 \setminus E} \|f(\xi_i)|I_i| - F(I_i)\| + \sum_{l, \xi \in E} \|F(I_i)\| + \sum_{l, \xi \in E} \|f(\xi_i)|I_i|\|
\]

\[
< \epsilon |I_0| + \sum_{i,j} \epsilon \cdot 2^{-(i+j)} + \sum_{i,j} j \cdot \eta_{i,j}
\]

\[
< \epsilon(2 + |I_0|)
\]

for arbitrary \(\delta\)-fine C-partition \(D_l = \{(I_i, \xi_i)\}\) of \(I_0\). Hence \(f\) is strongly C-integrable on \(I_0\).

\[\square\]

From Theorem 4.2 and Theorem 4.3 we get immediately the following theorem.

**Theorem 4.4.** A function \(f : I_0 \to X\) is strongly C-integrable on \(I_0\) if and only if there is a function \(F : I_0 \to X\) which is \(ACG^*_c\) on \(I_0\) such that \(f(\xi) = F'(\xi)\) almost everywhere on \(I_0\).

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**References**


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