Kangweon-Kyungki Math. Jour. **14** (2006), No. 2, pp. 169–183

ON C-INTEGRAL OF BANACH-VALUED FUNCTIONS

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Abstract. In this paper, we define and study the C-integral and the strong C-integral of functions mapping an interval [a,b] into a Banach space X. We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional, We also consider the property of primitives corresponding to Banach-valued functions with respect to the C-integral and the strong C-integral.

1. Introduction

It is well known that the Henstock integral is a kind of non-absolute integral and includes the Riemann, improper Riemann, Newton, and Lebesgue integrals. From the following function

$$F(x) = \begin{cases} x \sin \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases} \tag{1}$$

we know that it is a primitive for the Henstock integral, but it is neither a Lebesgue primitive, neither a differentiable function, nor a sum of Lebesgue primitive and a differentiable function. It is natural to ask that is there a minimal integral includes the Lebesgue integral and the derivative?

In 1996 B.Bongiorno provided a constructive minimal integration process of Riemann type, called C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable function.

Received August 23, 2006.
2000 Mathematics Subject Classification: 26A39, 28B05, 46G10.
Key words and phrases: C-integral, strong C-integral, the property S^*C.
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† Supported by NNSF of China (10571085) and by Science Foundation of Hohai University.

B.Bongiorno and L.Di Piazza [1, 2, 6] discussed some properties of the C-integral of real-valued functions.

In this paper, we define and study the C-integral and the strong C-integral of functions mapping an interval [a,b] into a Banach space X. We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional. If the function has the property S^*C then it is strongly C-integrable, but the converse is not true. We also consider the property of primitives corresponding to Banach-valued functions with respect to the C-integral and the strong C-integral.

2. Definitions and basic properties

Throughout this paper, $I_0 = [a, b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. A partition D is a finite collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of I_0. $\delta(\xi)$ is a positive function on I_0, i.e. $\delta(\xi) : I_0 \to \mathrm{R}^+$. We say that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is

(1) a partial partition of I_0 if $\bigcup_{i=1}^n I_i \subset I_0$,

(2) a partition of I_0 if $\bigcup_{i=1}^n I_i = I_0$,

(3) δ - fine McShane partition of I_0 if $I_i \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_0$ for all i=1,2,$\cdots$,n,

(4) δ - fine C-partition of I_0 if it is a δ - fine McShane partition of I_0 and satisfying the condition

$$\sum_{i=1}^{n} dist(\xi_i, I_i) < \frac{1}{\varepsilon}$$

for the given ϵ, here $dist(\xi_i, I_i) = inf\{|t_i - \xi_i| : t_i \in I_i\}$,

(5) δ - fine partition of I_0 if $\xi_i \in I_i \subset B(\xi_i, \delta(\xi_i))$ for all i=1,2,$\cdots$,n.

Given a δ - fine C-partition (McShane partition) $D = \{(I_i, \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)|I_i|$$

for integral sums over D, whenever $f : I_0 \to X$.

Definition 2.1. A function $f : I_0 \to X$ is C-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function

$\delta(\xi) : I_0 \to R^+$ such that

$$\|S(f, D) - A\| < \epsilon$$

for each δ - fine *C-partition* $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. A is called the *C-integral* of f on I_0, and we write $A = \int_{I_0} f$ or $A = \text{(C)} \int_{I_0} f$.

The function f is C-integrable on the set $E \subset I_0$ if the function $f\chi_E$ is C-integrable on I_0. We write$\int_E f = \int_{I_0} f\chi_E$.

We can easily get the following two theorems.

THEOREM 2.1. *A function $f : I_0 \to X$ is C-integrable if and only if for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$\|S(f, D_1) - S(f, D_2)\| < \epsilon$$

for arbitrary δ - fine C-partition D_1 and D_2 of I_0.

THEOREM 2.2. *Let $f : I_0 \to X$ and $g : I_0 \to X$.*

(1) if f is C-integrable on I_0, then f is C-integrable on every subinterval of I_0,

(2) if f is C-integrable on each of the intervals I_1 and I_2, where I_i are non-overlapping and $I_1 \bigcup I_2 = I_0$, then f is C-integrable on I_0 and $\int_{I_1} f + \int_{I_2} f = \int_{I_0} f$,

(3) if f and g are C-integrable on I_0 and if α and β are real numbers, then $\alpha f + \beta g$ is C-integrable on I_0 and $\int_{I_0}(\alpha f + \beta g) = \alpha \int_{I_0} f + \beta \int_{I_0} g$.

LEMMA 2.3. *(Saks-Henstock) Let $f : I_0 \to X$ is C-integrable on I_0. Then for each ε there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$\|S(f, D) - \int_{I_0} f\| < \epsilon$$

for each δ - fine C-partition $D = \{(I, \xi)\}$ of I_0. Particularly, if $D' = \{(I_i, \xi_i)\}_{i=1}^m$ is an arbitrary δ - fine partial C-partition of I_0, we have

$$\|S(f, D') - \sum_{i=1}^m \int_{I_i} f(\xi_i)\| \leq \epsilon.$$

Proof. The proof is similar to the case for Banach-valued Henstock integrable functions and the reader is referred to [7, Lemma 3.4.1.] for details. $\square$

THEOREM 2.4. *Let $f : I_0 \to X$, if $f = \theta$ almost everywhere on I_0, then f is C-integrable on I_0 and $\int_{I_0} f = \theta$.*

Proof. Let ϵ be given. Assume $E = \{\xi \in I_0 : f(\xi) \neq \theta\}$ and $E = \bigcup E_n \subset I_0$ where $E_n = \{\xi \in I_0 : n-1 \leq \|f(\xi)\| < n\}$. Obviously, $\mu(E) = 0$ and therefore $\mu(E_n) = 0$, then there are open sets $G_n \subset I_0$ such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{n \cdot 2^n}$. We define a positive function $\delta(\xi) : I_0 \to R^+$ in such a way that $\delta(\xi) = 1$ for $\xi \in I_0 \backslash E$ and $B(\xi, \delta(\xi)) \subset G_n$ if $\xi \in E_n$.

Suppose that $D = \{(I, \xi)\}$ is a δ - fine C-partition of I_0, then

$$\|\sum f(\xi)|I|\| \leq \sum n \cdot \frac{\epsilon}{n \cdot 2^n} \leq \epsilon.$$

Hence f is C-integrable on I_0 and $\int_{I_0} f = \theta$. □

COROLLARY 2.5. *Let $f : I_0 \to X$ is C-integrable on I_0. If $f = g$ almost everywhere on I_0, then g is C-integrable on I_0 and $\int_{I_0} f = \int_{I_0} g$ almost everywhere on I_0.*

THEOREM 2.6. *Let $f : I_0 \to X$ is C-integrable on I_0.*

(1) for each $x^* \in X^*$, the function $x^* f$ is C-integrable on I_0 and $\int_{I_0} x^* f = x^*(\int_{I_0} f)$.

(2) If $T : X \to Y$ is a continuous linear operator, then Tf is C-integrable on I_0 and $\int_{I_0} Tf = T(\int_{I_0} f)$.

Proof. (1) Since $f : I_0 \to X$ is C-integrable on I_0, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to \mathrm{R}^+$ such that

$$\|S(f, D) - \int_{I_0} f\| < \frac{\epsilon}{\|x^*\|}$$

for each δ - fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence for each $x^* \in X^*$, we have

$$||S(x^* f, D) - x^*(\int_{I_0} f)\| \leq \|x^*\| \cdot \|S(f, D) - \int_{I_0} f\| < \epsilon.$$

(2) $T : X \to Y$ is a continuous linear operator, then there exists a number $M > 0$ such that $\|Tx\| \leq M\|x\|$ for each $x \in X$. Since $f : I_0 \to X$ is C-integrable on I_0, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|S(f, D) - \int_{I_0} f\| < \frac{\epsilon}{M}$$

for each δ - fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence we have

$$\|S(Tf, D) - T(\int_{I_0} f)\| \leq M \cdot \|S(f, D) - \int_{I_0} f\| < \epsilon.$$

$\square$

3. C-integral and strong C-integral

Definition 3.1. A function $f : I_0 \to X$ is strongly C-integrable if there exists an additive function $F : I_0 \to X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\sum_{i=1}^{n} \|f(\xi_i)|I_i| - F(I_i)\| < \epsilon$$

for each δ - fine C-partition $D = \{(I_i, \xi_i)\}_{i=1}^{n}$ of I_0. $F(I_0)$ is called the strong C-integral of f on I_0, and we write $F(I_0) = \int_{I_0} f$.

Theorem 3.1. *Let $f : I_0 \to X$ is strongly C-integrable on I_0, then f is C-integrable on I_0.*

Proof. It follows from the definitions of the strong C-integral and C-integral that if f is strongly C-integrable on I_0, then f is C-integrable on I_0. $\square$

Definition 3.2. A function $f : I_0 \to X$ is strongly Henstock (McShane) integrable if there exists an additive function $F : I_0 \to X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\sum_{i=1}^{n} \|f(\xi_i)|I_i| - F(I_i)\| < \epsilon$$

for each δ - fine (McShane) partition $D = \{(I_i, \xi_i)\}_{i=1}^{n}$ of I_0. $F(I_0)$ is called the strong Henstock (McShane) integral of f on I_0, and we write $F(I_0) = \int_{I_0} f$.

By the definitions of strong Henstock, strong C-integral and strong McShane integral and the fact that each δ - fine partition is also δ - fine C-partition and therefore is also δ - fine McShane partition, we get immediately the following Theorem.

THEOREM 3.2. *Let the function $f : I_0 \to X$.*
(a) if f is strongly McShane integrable then f is strongly C-integrable.
(b) if f is strongly C-integrable then f is strongly Henstock integrable.

THEOREM 3.3. *If a Banach space X is finite dimensional then a function $f : I_0 \to X$ is C-integrable on I_0 if and only if f is strongly C-integrable on I_0.*

Proof. We only prove if X is finite dimensional and $f : I_0 \to X$ is C-integrable on I_0 then f is strongly C-integrable on I_0.

Let ϵ be given. It follows from the definition of the C-integral that there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|\sum f(\xi)|I| - F(I)\| < \epsilon$$

for each δ -fine C-partition $D = \{(I, \xi)\}$ of I_0. Let $\{e_1, e_2, \cdots, e_n\}$ be a base of X and $g_i : I_0 \to R$ $(i = 1, 2, \cdots, n)$. By the Hahn-Banach Theorem, for each e_i there is $x_i^* \in x^*$ such that

$$x_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2}$$

for $i, j = 1, 2, \cdots, n$ and therefore $x_i^*(f) = \sum_{j=1}^n g_j x_i^*(e_j) = g_i$. Since $g_i : I_0 \to R$ is C-integrable on I_0 from Theorem 2.6, there is a positive function $\delta_i(\xi) : I_0 \to R^+$ such that

$$|S(g_i, D_i) - \sum \int_I g_i| < \epsilon$$

for each δ_i - fine C-partition $D_i = \{(I, \xi)\}$ of I_0. By an easy adaptation of Saks-Henstock Lemma we have

$$\sum |g_i(\xi)|I| - \int_I g_i| < 2\epsilon.$$

We also have

$$F(I) = \int_I f = \int_I \sum_{i=1}^n g_i e_i = \sum_{i=1}^n \int_I g_i e_i = \sum_{i=1}^n e_i G_i(I)$$

where $G_i(I) = \int_I g_i$. Let $\delta(\xi) < \delta_i(\xi)$ for $i = 1, 2, \cdots, n$ and consequently

$$\begin{aligned}\sum \|f(\xi)|I| - F(I)\| &= \sum \|\sum_{i=1}^{n} g_i(\xi)e_i|I| - \sum_{i=1}^{n} e_i G_i(I)\| \\ &\leq \sum_{i=1}^{n} \|e_i\| \sum |g_i(\xi)|I| - G_i(I)| \\ &< \epsilon \cdot \sum_{i=1}^{n} \|e_i\|\end{aligned}$$

for each δ - fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence f is strongly C-integrable on I_0. □

THEOREM 3.4. *If X is a Banach space then the C-integral is equivalent to the strong C-integral on I_0 if and only if X is finite dimensional.*

Proof. The proof of necessity is similar to the case for Henstock (McShane) integral, see, for example, [8, Theorem 3]. In [8], if X is infinite dimensional, there exist $x_1^r, x_2^r, \cdots, x_{2^r}^r \in X$ such that

$$\|x_i^r\| = \frac{1}{2^r}$$

for each r, $1 \leq i \leq 2^r$ and we also have

$$\|\sum_{i=1}^{2^r} \theta_i^r x_i^r\|^2 \leq \frac{3}{2^r}$$

for every θ_i^r with $|\theta_i^r| \leq 1$, $1 \leq i \leq 2^r$. V.A.Svortsov and A.P.Solodov defined a function $f : [0, 1] \to X$ in the following way

$$f(t) = \begin{cases} 0 & \text{if } t \in C, \text{ or } t = d_i^r, r \geq 0, 1 \leq i \leq 2^r, \\ 2 \cdot 3^r x_i^r & \text{if } t \in (a_i^r, d_i^r), r \geq 0, 1 \leq i \leq 2^r, \\ -2 \cdot 3^r x_i^r & \text{if } t \in (d_i^r, b_i^r), r \geq 0, 1 \leq i \leq 2^r. \end{cases} \tag{3}$$

Here C be the cantor ternary set, $(a_i^r, b_i^r), r \geq 0, 1 \leq i \leq 2^r$ being the intervals of rank r contiguous to C with middle points d_i^r and satisfying $b_i^r - a_i^r = 3^{-(r+1)}$.

The function is McShane integrable but is not strongly Henstock integrable, then we have f is C-integrable and is not strongly C-integrable.

In other words, if the C-integral is equivalent to the strong C-integral then X is finite dimensional. □

Definition 3.3. A function $F : I_0 \to X$ is differentiable at $\xi \in I_0$ if there is a $f(\xi) \in X$ such that

$$\lim_{\delta \to 0} \| \frac{F(\xi + \delta) - F(\xi)}{\delta} - f(\xi) \| = 0.$$

We denote $f(\xi) = F'(\xi)$ the derivative of F at ξ.

Theorem 3.5. *If a function $F : I_0 \to X$ is differentiable on I_0 with $f(\xi) = F'(\xi)$ for each $\xi \in I_0$, then $f : I_0 \to X$ is strongly C-integrable.*

Proof. Let ϵ be given. By the definition of derivative, for each $\xi \in I_0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\| \frac{F(\zeta) - F(\xi)}{\zeta - \xi} - f(\xi) \| < \frac{\epsilon^2}{2(1 + \epsilon |I_0|)}$$

for all $\zeta \in I_0$ with $|\zeta - \xi| < \delta(\xi)$. Assume $D = \{(I_i, \xi_i)\}_{i=1}^n$ is an arbitrary δ - fine C-partition of I_0, we have

$$\begin{aligned} \sum_{i=1}^n \| f(\xi_i)|I_i| - F(I_i) \| &< \frac{\epsilon^2}{1 + \epsilon |I_0|} \sum_{i=1}^n (dist(\xi_i, I_i) + |I_i|) \\ &< \frac{\epsilon^2}{1 + \epsilon |I_0|} (\frac{1}{\epsilon} + |I_0|) < \epsilon. \end{aligned}$$

Hence $f : I_0 \to X$ is strongly C-integrable on I_0. □

Definition 3.4. A function $f : I_0 \to X$ has the property S^*C if for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\sum_{i=1}^m \sum_{j=1}^n \| f(\xi_i) - f(\zeta_j) \| \cdot |I_i \bigcap L_j| < \epsilon$$

for arbitrary δ - fine C-partitions $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of I_0.

Theorem 3.6. *If a function $f : I_0 \to X$ has the property S^*C then f is strongly C-integrable on I_0*

Proof. We will prove the Theorem in two steps.

Step 1. Let ϵ be given. Assume $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ are arbitrary δ - fine C-partition of I_0 then we have

$$\begin{aligned} & \|\sum_{i=1}^m f(\xi_i)|I_i| - \sum_{j=1}^n f(\zeta_i)|L_j|\| \\ = & \|\sum_{j=1}^n \sum_{i=1}^m f(\xi_i)|I_i \bigcap L_j| - \sum_{i=1}^m \sum_{j=1}^n f(\zeta_j)|I_i \bigcap L_j|\| \\ = & \|\sum_{i=1}^m \sum_{j=1}^n (f(\xi_i) - f(\zeta_i))|I_i \bigcap L_j|\| \\ \leq & \sum_{i=1}^m \sum_{j=1}^n \|f(\xi_i) - f(\zeta_i)\| \cdot |I_i \bigcap L_j| < \epsilon. \end{aligned}$$

By Theorem 2.1 we have f is C-integrable on I_0.

Step 2. By Definition 3.4, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \rightarrow R^+$ such that

$$\sum_{i=1}^m \sum_{j=1}^n \|f(\xi_i) - f(\zeta_j)\| \cdot |I_i \bigcap L_j| < \epsilon$$

for arbitrary δ - fine C-partition $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of I_0. f is C-integrable on I_0 and therefore C-integrable on I_i for $i = 1, 2, \cdots, m$. Hence for given $\varepsilon > 0$ there is a positive function $\delta_i(\xi) : I_0 \rightarrow R^+$ such that $\delta_i(\xi) \leq \delta(\xi)$ and such that for any δ_i -fine C-partition $D_i = \{(L_j^i, \zeta_j^i)\}_{j=1}^{n^i}$ for $i = 1, 2, \cdots, m$ of I_i, then

$$\|\sum_{j=1}^{n^i} f(\zeta_j^i)|L_j^i| - \int_{I_i} f\| = \|\sum_{j=1}^{n^i} [f(\zeta_j^i)|L_j^i| - \int_{L_j^i} f]\| < \frac{\epsilon}{2m}.$$

Assume $D = \bigcup_{i=1}^{m} D_i$, obviously it is a δ -fine C-partition of I_0. Hence we have

$$\begin{aligned}
&\sum_{i=1}^{m} \|f(\xi_i)|I_i| - F(I_i)\| \\
= &\sum_{i=1}^{m} \| \sum_{j=1}^{n^i} f(\xi_i)|I_i \bigcap L_j^i| - \sum_{j=1}^{n^i} F(I_i \bigcap L_j^i)\| \\
= &\sum_{i=1}^{m} \| \sum_{j=1}^{n^i} (f(\xi_i) - f(\zeta_j^i))|I_i \bigcap L_j^i| + \sum_{j=1}^{n^i} [f(\zeta_j^i)|I_i \bigcap L_j^i| - F(I_i \bigcap L_j^i)]\| \\
\leq &\sum_{i=1}^{m} \sum_{j=1}^{n^i} \|f(\xi_i) - f(\zeta_j^i)\| \cdot |I_i \bigcap L_j^i| \\
&\qquad + \sum_{i=1}^{m} \| \sum_{j=1}^{n^i} [f(\zeta_j^i)|I_i \bigcap L_j^i| - F(I_i \bigcap L_j^i)]\| \\
< &\frac{\epsilon}{2} + \sum_{i=1}^{m} \frac{\epsilon}{2m} = \epsilon.
\end{aligned}$$

By Definition 3.1 , f is strongly C-integrable on I_0. □

REMARK 3.1. The converse of Theorem 3.6 is not true, in other words, if f is strongly C-integrable on I_0, it does not necessarily have the property S^*C.

Proof. Assume a real-valued function f by

$$f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{4}$$

It is easy to know that the primitive of f is

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{5}$$

$F(x)$ is differentiable everywhere and $F'(x) = f(x)$ everywhere on $[0, 1]$, then $f(x)$ is strongly C-integrable on $[0, 1]$ from Theorem 3.5.

Assume f has the property S^*C, then for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [0,1] \to R^+$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} |f(\xi_i) - f(\zeta_j)| \cdot |I_i \bigcap L_j| < \epsilon$$

for arbitrary δ - fine C-partitions $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of $[0,1]$. Consequently we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ||f(\xi_i)| - |f(\zeta_j)|| \cdot |I_i \bigcap L_j| \le \sum_{i=1}^{m}\sum_{j=1}^{n} |f(\xi_i) - f(\zeta_j)| \cdot |I_i \bigcap L_j| < \epsilon$$

and therefore $|f(x)|$ has the property S^*C, then $|f(x)|$ is C-integrable on $[0,1]$ from Theorem 3.6 and Theorem 3.1. We claim that $f(x)$ is Lebesgue (McShane) integrable, the proof is too easy and will be omitted. But in fact $F(x)$ is not absolutely continuous on $[0,1]$ and therefore $f(x)$ is not Lebesgue integrable on $[0,1]$, then $f(x)$ does not have the property S^*C. $\square$

4. Primitive of the C-integral and strong C-integral

Definition 4.1. Let $F : I_0 \to X$ and let E be a subset of I_0.

(a)F is said to be ${}^*AC_c$ on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\| \sum_i F(I_i)\| < \epsilon$ for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying $\xi_i \in E$ for each i and $\sum_i |I_i| < \eta$.

(b)F is said to be AC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\sum_i \|F(I_i)\| < \epsilon$ for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying $\xi_i \in E$ for each i and $\sum_i |I_i| < \eta$.

(c)F is said to be AC_c^* on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\sum_i \omega(F, I_i) < \epsilon$ for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying $\xi_i \in E$ for each i and $\sum_i |I_i| < \eta$.

(d)F is said to be ${}^*ACG_c$ (ACG_c, ACG_c^*)on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is ${}^*AC_c$ (AC_c, AC_c^*).

Theorem 4.1. *If a function $f : I_0 \to X$ is C-integrable on I_0 with the primitive $F : I_0 \to X$, then F is ${}^*ACG_c$ on I_0*

Proof. Since f is C-integrable on I_0, by the Saks-Henstock Lemma, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|\sum_{i=1}^{m}[f(\xi_i)|I_i| - F(I_i)]\| \leq \epsilon$$

for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}_{i=1}^{m}$ of I_0.

Let $E_n = \{\xi \in I_0 : n - 1 \leq \|f(\xi)\| < n\}$ for $n = 1, 2, \cdots$. Obviously, the union of E_n is the whole interval I_0. Assume that $\xi_i \in E_n$ for each i and $\sum_i^m |I_i| < \frac{\epsilon}{n}$. Then we have

$$\begin{aligned} \|\sum_{i}^{m} F(I_i)\| &\leq \|\sum_{i=1}^{m}[F(I_i) - f(\xi_i)|I_i|]\| + \|\sum_{i=1}^{m} f(\xi_i) \cdot |I_i|\| \\ &\leq \epsilon + n \cdot \sum_{i=1}^{m} |I_i| < 2\epsilon. \end{aligned}$$

This shows that F is $^*AC_c$ on E_n. Hence F is $^*ACG_c$ on I_0.

□

THEOREM 4.2. *If a function $f : I_0 \to X$ is strongly C-integrable on I_0 with the primitive $F : I_0 \to X$, then F is ACG_c^* on I_0.*

Proof. Since f is strongly C-integrable on I_0, by the Saks-Henstock Lemma, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\sum_{i=1}^{m} \|f(\xi_i)|I_i| - F(I_i)\| \leq \epsilon$$

for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}_{i=1}^{m}$ of I_0.

It is easy to get that the primitive F is continuous on I_0. Thus, for each $i = 1, 2, ..., m$ there exists $I_i^{'} \subset I_i$ such that the oscillation $\omega(F, I_i)$ of F on I_i equals $\|F(I_i^{'})\|$, i.e., $\omega(F, I_i) = \|F(I_i^{'})\|$. Consequently, we can get a new δ - fine partial C-partition $D_0 = \{(I_i^{'}, \xi_i)\}_{i=1}^{m}$ of I_0. Using the Saks-Henstock Lemma again, we have

$$\sum_{i=1}^{m} \|f(\xi_i)|I_i^{'}| - F(I_i^{'})\| \leq \epsilon.$$

Let $E_n = \{\xi \in I_0 : n-1 \leq \|f(\xi)\| < n\}$ for $n = 1, 2, \cdots$. Obviously, $I_0 = \bigcup_n E_n$. Assume that $\xi_i \in E_n$ for each i and $\sum_i^m |I_i'| < \frac{\epsilon}{n}$. We have

$$\begin{aligned} \sum_{i=1}^m \omega(F, I_i) &= \sum_{i=1}^m \|F(I_i')\| \\ &\leq \sum_{i=1}^m \|F(I_i') - f(\xi_i)|I_i'|\| + \sum_{i=1}^m \|f(\xi_i)\| \cdot |I_i'| \\ &\leq \epsilon + n \cdot \sum_{i=1}^m |I_i'| < 2\epsilon. \end{aligned}$$

Hence F is AC_c^* on E_n. Consequently, F is ACG_c^* on I_0. □

THEOREM 4.3. *If a function $F : I_0 \to X$ is ACG_c^* on I_0 and $f(\xi) = F'(\xi)$ almost everywhere on I_0, then $f : I_0 \to X$ is strongly C-integrable on I_0 with the primitive $F : I_0 \to X$.*

Proof. There exists a set $E \subset I_0$ be of measure zero such that $f(\xi) = F'(\xi)$ for $\xi \in I_0 \backslash E$. For $\xi \in I_0 \backslash E$, by the definition of derivative, for each $\varepsilon > 0$, there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|f(\xi)|I| - F(I)\| < \epsilon |I|$$

for $|I| < \delta(\xi)$.

F is ACG_c^*, then there is a sequence of closed set $\{E_i\}$ such that $I_0 = \bigcup_i E_i$ and F is AC^* on each E_i. Let $E_{i,j} = \{\xi \in E \cap Y_i : j-1 \leq \|f(\xi)\| < j\}$ where $Y_1 = E_1, Y_i = E_i \backslash (E_1 \cup E_2 \cup \cdots \cup E_{i-1})$ for $i = 2, 3, \cdots$. Obviously $E_{i,j}, j = 1, 2, \cdots$ are pairwise disjoint and $E = \bigcup E_{i,j}$. F is also AC_c^* on E_{ij}, So there is a $\eta_{i,j} < \frac{\epsilon}{j \cdot 2^{i+j}}$ such that

$$\sum_k \|F(I_k)\| < \epsilon \cdot 2^{-(i+j)}$$

for arbitrary δ - fine partial C-partition $D = \{(I_k, \xi_k)\}$ satisfying at least one endpoint of I_k belonging to $E_{i,j}$ and $\sum_k |I_k| < \eta_{i,j}$.

There are open intervals G_{ij} such that $E_{i,j} \subset G_{i,j}$ and $|G_{i,j}| < \eta_{i,j}$. Now for $\xi \in E_{i,j}, i = 1, 2, \cdots$, put $\delta(\xi) > 0$ such that $B(\xi_i, \delta(\xi_i)) \subset G_{i,j}$. Hence we have defined a positive function $\delta(\xi)$ on I_0. Splitting the sum

$\sum$ into two partial sums with $\xi \in E$ and $\xi \in I_0 \backslash E$ respectively, we have

$$\begin{aligned}
&\sum_l \|f(\xi_l)|I_l| - F(I_l)\| \\
= &\sum_{l,\xi \in I_0 \backslash E} \|f(\xi_l)|I_l| - F(I_l)\| + \sum_{l,\xi \in E} \|f(\xi_l)|I_l| - F(I_l)\| \\
\leq &\sum_{l,\xi \in I_0 \backslash E} \|f(\xi_l)|I_l| - F(I_l)\| + \sum_{l,\xi \in E} \|F(I_l)\| + \sum_{l,\xi \in E} \|f(\xi_l)|I_l|\| \\
< &\ \epsilon|I_0| + \sum_{i,j} \epsilon \cdot 2^{-(i+j)} + \sum_{i,j} j \cdot \eta_{i,j} \\
< &\ \epsilon(2 + |I_0|)
\end{aligned}$$

for arbitrary δ -fine C-partition $D_l = \{(I_l, \xi_l)\}$ of I_0. Hence f is strongly C-integrable on I_0.

□

From Theorem 4.2 and Theorem 4.3 we get immediately the following theorem.

THEOREM 4.4. *A function $f : I_0 \to X$ is strongly C-integrable on I_0 if and only if there is a function $F : I_0 \to X$ which is ACG_c^* on I_0 such that $f(\xi) = F'(\xi)$ almost everywhere on I_0.*

Acknowledgement. The authors are grateful to the referee for his or her careful reading of the manuscript and for valuable and helpful suggestions.

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