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RELATIONS OF SHORT EXACT SEQUENCES CONCERNING AMALGAMATED FREE PRODUCTS

Woo Taeg Shin

Abstract. In this paper, we investigate the mutual relation among short exact sequences of amalgamated free products which involve augmentation ideals and relation modules. In particular, we find out commutative diagrams having a steady structure in the sense that all of their three columns and rows are short exact sequences.

1. Introduction

Let $\wp_1$ and $\wp_2$ be group presentations for H and K, respectively and $\wp$ presentation for $G = H *_U K$, i.e., the amalgamated free product of H and K with a subgroup U. It is known that short exact sequences of amalgamated free products are closely related. We can find out the relation among them by applying diagrams of groups(modules).

In this paper, we investigate the mutual relation among short exact sequences of amalgamated free products which involve augmentation ideals and relation modules. In particular, through the following main theorem, we find out commutative diagrams having a steady structure in the sense that all of their three columns and rows are short exact sequences. As a consequence of the main theorem, we have the corollary, which shows the evident relation, that is to say, necessary and sufficient conditions between (1-1) and (1-2).

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THEOREM 1.1. *For $G = H *_U K$, we have the following commutative diagrams :*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathbb{Z}G \otimes_U IU & \stackrel{\alpha_1}{\rightarrow} & (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) & \stackrel{\beta_1}{\rightarrow} & IG & \rightarrow & 0 \\
 & & \downarrow^{\iota'} & & \downarrow^{\iota^*} & & \downarrow^{\iota} & & \\
0 & \rightarrow & \mathbb{Z}G \otimes_U \mathbb{Z}U & \stackrel{\alpha_2}{\rightarrow} & (\mathbb{Z}G \otimes_H \mathbb{Z}H) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}K) & \stackrel{\beta_2}{\rightarrow} & \mathbb{Z}G & \rightarrow & 0 \\
 & & \downarrow^{\varepsilon'} & & \downarrow^{\varepsilon^*} & & \downarrow^{\varepsilon} & & \\
0 & \rightarrow & \mathbb{Z}G \otimes_U \mathbb{Z} & \stackrel{\alpha_3}{\rightarrow} & (\mathbb{Z}G \otimes_H \mathbb{Z}) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}) & \stackrel{\beta_3}{\rightarrow} & \mathbb{Z} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where IU, IH, IK, and IG are the augmentation ideals of $\mathbb{Z}U$, $\mathbb{Z}H$, $\mathbb{Z}K$, and $\mathbb{Z}G$ respectively.

COROLLARY 1.2. *(1-1) is short exact if and only if (1-2) is short exact.*
$(1-1)$ $0 \longrightarrow \mathbb{Z}G \otimes_U IU \xrightarrow{\alpha_1} (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) \xrightarrow{\beta_1} IG \longrightarrow 0.$
$(1-2)$ $0 \longrightarrow \mathbb{Z}G \otimes_U \mathbb{Z} \xrightarrow{\alpha_3} (\mathbb{Z}G \otimes_H \mathbb{Z}) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}) \xrightarrow{\beta_3} \mathbb{Z} \longrightarrow 0.$

2. Preliminaries

In this section we have some basic facts on short exact sequences, which will be useful for our purpose. Suppose that we have a sequence $\{G_n\}$ of groups(modules) and a sequence of group(module) homomorphisms f_i from G_i into G_{i+1}. We will express these homomorphisms by arrows between the groups(modules):

$$(2-1) \qquad \cdots \longrightarrow G_{n-1} \xrightarrow{f_{n-1}} G_n \xrightarrow{f_n} G_{n+1} \longrightarrow \cdots$$

The set of suffixes may be finite or infinite. The above sequence (2-1) is said to be *exact* if we have $im\ f_{n-1} = ker\ f_n$ for each n. If $G_i = 0$ for $i \leq n-2$ and $G_i = 0$ for $i \geq n+2$, then

$$(2-2) \qquad 0 \longrightarrow G_{n-1} \longrightarrow G_n \longrightarrow G_{n+1} \longrightarrow 0.$$

The sequence (2-2) is called a *short exact sequence.*

Let $A, B, C,$ and D be groups(modules) and let α,β,γ, and δ be group(module) homomorphisms. We say that the *diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow^{\gamma} & & \downarrow^{\beta} \\ C & \xrightarrow{\delta} & D \end{array}$$

is *commutative* if $\beta\alpha = \delta\gamma : A \longrightarrow D$. This notion can be generalized to more complicated diagrams in an obvious way.

LEMMA 2.1. *Consider the following commutative diagram, where three columns are exact.*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Suppose that the middle row is exact. Then the first row is exact if and only if the third row is exact.

Let H,K, and U be groups and ϕ_1 and ϕ_2 homomorphisms:

$$(2-3) \qquad \begin{array}{ccc} U & \xrightarrow{\phi_1} & H \\ {}^{\phi_2}\downarrow & & \\ K & & \end{array}$$

A *solution* of the above diagram (2-3) is a group G and homomorphisms ψ_1 and ψ_2 such that the following diagram commutes (i.e., $\psi_1\phi_1 = \psi_2\phi_2$):

$$(2-4) \qquad \begin{array}{ccc} U & \xrightarrow{\phi_1} & H \\ {}^{\phi_2}\downarrow & & {}^{\psi_1}\downarrow \\ K & \xrightarrow{\psi_2} & G \end{array}$$

A *push-out* of the diagram (2-3) is a solution (G, ψ_1, ψ_2) such that, for any other solution (L, θ_1, θ_2), there exists a unique homomorphism $\alpha : G \longrightarrow L$ such that $\theta_i = \alpha\psi_i$ $(i = 1,\ 2)$. As usual, the push-out is unique up to isomorphism.

Let $\wp = \langle\, \mathbf{x} : \mathbf{r} \,\rangle$ be a group presentation, where $\mathbf{x}$ is a set and $\mathbf{r}$ is a set of cyclically reduced words on $\mathbf{x} \cup \mathbf{x}^{-1}$. Let N be the normal closure of $\mathbf{r}$ in F, where F is the free group on $\mathbf{x}$. Then the quotient G of F by N is called the *group defined by* $\wp$.

THEOREM 2.2. *A push-out exists for the diagram (2-3). Moreover, if H and K are defined by $\wp_1 = \langle\, \mathbf{x}_1 : \mathbf{r}_1 \,\rangle$ and $\wp_2 = \langle\, \mathbf{x}_2 : \mathbf{r}_2 \,\rangle$ respectively, then the push-out G is defined by $\wp = \langle\, \mathbf{x}_1 \cup \mathbf{x}_2 : \mathbf{r}_1 \cup \mathbf{r}_2 \cup \{\phi_1(u)\phi_2(u)^{-1} : u \in U\} \,\rangle$.*

A proof of this theorem can be found in [12] (Theorem 11.58). When both ϕ_1 and ϕ_2 are monomorphisms, the push-out G is called the *amalgamated free product* of H and K with a subgroup U. In this case we usually regard U as a subgroup of H and K, and regard ϕ_1 and ϕ_2 as inclusions. The usual notation for the amalgamated free product of H and K with a subgroup U is $H *_U K$. Sometimes it is more convenient to use the notation $H *_{U \cong V} K$ where $U \subseteq H$, $V \subseteq K$, and $U \cong V$. For more precision, we could mention the specific isomorphism from U to V. For an amalgamated free product we see that ψ_1 and ψ_2 are monomorphisms, and we regard them as inclusions.

3. Main results

Let G be a group written multiplicatively. The *integral group ring* $\mathbb{Z}G$ of G is defined as follows. Its underlying abelian group is the free abelian group on the set of elements of G as basis ; the product of two basis elements is given by the product in G. Thus the elements of the group ring $\mathbb{Z}G$ are sums $\sum_{x \in G} m(x)x$ where m is a function from G to $\mathbb{Z}$ which takes the value zero except on a finite number of elements of G. The multiplication is given by $(\sum_{x \in G} m(x)x) \cdot (\sum_{y \in G} m'(y)y) = \sum_{x,y \in G} (m(x) \cdot m'(y))xy$. The group ring is characterized by the following universal property. Let $i : G \longrightarrow \mathbb{Z}G$ be the obvious embedding.

PROPOSITION 3.1. *Let R be a ring. To each function $f : G \longrightarrow R$ such that $f(xy) = f(x) \cdot f(y)$ and $f(1) = 1_R$, there exists a unique ring homomorphism $f' : \mathbb{Z}G \longrightarrow R$ such that $f'i = f$.*

A (left) G-module is an abelian group A together with a group homomorphism $\sigma : G \longrightarrow AutA$. In other words, each element of G acts as an automorphism of A. Since $AutA \subseteq EndA$, the universal property of the group ring yields a ring homomorphism $\sigma' : \mathbb{Z}G \longrightarrow EndA$, making A into a (left) module over $\mathbb{Z}G$. Conversely, if A is a (left) module over $\mathbb{Z}G$ then A is a (left) G-module, since any ring homomorphism takes invertible elements into invertible elements, and since the group elements in $\mathbb{Z}G$ are invertible. Thus we need not retain any distinction between the concepts of G-module and $\mathbb{Z}G$-module. A (left) G-module is called *trivial* if the structure map $\sigma : G \longrightarrow AutA$ is trivial, i.e., if every element of G acts as the identity in A. Every abelian group may be regarded as a trivial left or right G-module for each group G. We regard $\mathbb{Z}$ as a left $\mathbb{Z}G$-module with the trivial G-action. The *augmentation map* $\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$ is the homomorphism sending every $x \in G$ into $1 \in \mathbb{Z}$, that is $\sum_{x\in G} m(x)x \longmapsto \sum_{x\in G} m(x)$. The kernel of ε is denoted by IG and is called the *augmentation ideal* of $\mathbb{Z}G$. Thus we have a short exact sequence

$$(3-1) \qquad 0 \longrightarrow IG \xrightarrow{\iota} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Tensoring (3-1) with IG over$\mathbb{Z}$, we obtain the short exact sequence

$$(3-2) \qquad 0 \longrightarrow IG \otimes_{\mathbb{Z}} IG \xrightarrow{\gamma} \mathbb{Z}G \otimes_{\mathbb{Z}} IG \xrightarrow{\delta} IG \longrightarrow 0$$

where γ and δ are defined by

$$\gamma : (x-1) \otimes (x-1) \longmapsto (x-1) \otimes (x-1) \ (x \in G)$$
$$\delta : x \otimes (y-1) \longmapsto x(y-1) \ (x, y \in G).$$

Let $G = H *_U K$ be the amalgamated free product of H and K with subgroup U. Then we have:

PROPOSITION 3.2. *There is a short exact sequence*

$$(3-3) \quad 0 \longrightarrow \mathbb{Z}G \otimes_U IU \xrightarrow{\alpha_1} (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) \xrightarrow{\beta_1} IG \longrightarrow 0$$

where α_1 *and* β_1 *are defined by*
$\alpha_1 : x \otimes (u-1) \longmapsto (x \otimes (u-1), -x \otimes (u-1))$ $(x \in G,\ u \in U)$
$\beta_1 : (x \otimes (h-1), y \otimes (k-1)) \longmapsto x(h-1) + y(k-1)$ $(x, y \in G, h \in H, k \in K)$.

PROPOSITION 3.3. *There is a short exact sequence*

$$(3-4) \quad 0 \longrightarrow \mathbb{Z}G \otimes_U \mathbb{Z}U \xrightarrow{\alpha_2} (\mathbb{Z}G \otimes_H \mathbb{Z}H) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}K) \xrightarrow{\beta_2} \mathbb{Z}G \longrightarrow 0$$

where α_2 *and* β_2 *are defined by*

$$\begin{aligned}\alpha_2 &: x\otimes u \longmapsto (x\otimes u, -x\otimes u)\ (x\in G,\ u\in U)\\ \beta_2 &: (x\otimes h, y\otimes k) \longmapsto xh+yk\ (x,y\in G,\ h\in H,\ k\in K).\end{aligned}$$

Proposition 3.4. *There is a short exact sequence*

$$(3-5)\qquad 0\longrightarrow \mathbb{Z}G\otimes_U \mathbb{Z} \xrightarrow{\alpha_3} (\mathbb{Z}G\otimes_H \mathbb{Z})\oplus(\mathbb{Z}G\otimes_K \mathbb{Z}) \xrightarrow{\beta_3} \mathbb{Z}\longrightarrow 0$$

where α_3 *and* β_3 *are defined by*

$$\begin{aligned}\alpha_3 &: x\otimes a \longmapsto (x\otimes a, -x\otimes a)\ (x\in G,\ a\in \mathbb{Z})\\ \beta_3 &: (x\otimes a, y\otimes b) \longmapsto a+b\ (x,y\in G,\ a,b\in \mathbb{Z}).\end{aligned}$$

We now observe the relation among (3-3),(3-4), and (3-5) through the following theorem. Then we can find out commutative diagrams having a steady structure.

Theorem 3.5. *The following diagram is commutative :*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z}G\otimes_U IU & \xrightarrow{\alpha_1} & (\mathbb{Z}G\otimes_H IH)\oplus(\mathbb{Z}G\otimes_K IK) & \xrightarrow{\beta_1} & IG & \to & 0\\
 & & \downarrow{\iota'} & & \downarrow{\iota^*} & & \downarrow{\iota} & & \\
0 & \to & \mathbb{Z}G\otimes_U \mathbb{Z}U & \xrightarrow{\alpha_2} & (\mathbb{Z}G\otimes_H \mathbb{Z}H)\oplus(\mathbb{Z}G\otimes_K \mathbb{Z}K) & \xrightarrow{\beta_2} & \mathbb{Z}G & \to & 0\\
 & & \downarrow{\varepsilon'} & & \downarrow{\varepsilon^*} & & \downarrow{\varepsilon} & & \\
0 & \to & \mathbb{Z}G\otimes_U \mathbb{Z} & \xrightarrow{\alpha_3} & (\mathbb{Z}G\otimes_H \mathbb{Z})\oplus(\mathbb{Z}G\otimes_K \mathbb{Z}) & \xrightarrow{\beta_3} & \mathbb{Z} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where
$\iota' : x\otimes(u-1) \longmapsto x\otimes(u-1)\ (x\in G,\ u\in U)$
$\varepsilon' : x\otimes u \longmapsto x\otimes 1\ (x\in G,\ u\in U)$
$\iota^* : (x\otimes(h-1), y\otimes(k-1)) \longmapsto (x\otimes(h-1), y\otimes(k-1))\ (x,y\in G,\ h\in H,\ k\in K)$
$\varepsilon^* : (x\otimes h, y\otimes k) \longmapsto (x\otimes 1, y\otimes 1)\ (x,y\in G,\ h\in H,\ k\in K)$

Proof. (1) We consider the commutativity of the left upper hand square. Then
$\iota^*\alpha_1(x\otimes(u-1)) = \iota^*(x\otimes(u-1), -x\otimes(u-1)) = (x\otimes(u-1), -x\otimes(u-1))$
$\alpha_2\iota'(x\otimes(u-1)) = \alpha_2(x\otimes(u-1)) = (x\otimes(u-1), -x\otimes(u-1))$.
Thus we have $\iota^*\alpha_1 = \alpha_2\iota'$. Hence the left upper hand square is commutative.
(2) We consider the commutativity of the right upper hand square. Then

$\iota\beta_1(x\otimes(h-1), y\otimes(k-1)) = \iota(x(h-1)+y(k-1)) = x(h-1)+y(k-1)$
$\beta_2\iota^*(x\otimes(h-1), y\otimes(k-1)) = \beta_2(x\otimes(h-1), y\otimes(k-1)) = x(h-1)+y(k-1).$
Thus we have $\iota\beta_1 = \beta_2\iota^*$. Hence the right upper hand square is commutative.
(3) We consider the commutativity of the left lower hand square. Then

$$\varepsilon^*\alpha_2(x\otimes u) = \varepsilon^*(x\otimes u, -x\otimes u) = (x\otimes 1, -x\otimes 1)$$
$$\alpha_3\varepsilon'(x\otimes u) = \alpha_3(x\otimes 1) = (x\otimes 1, -x\otimes 1).$$

Thus we have $\varepsilon^*\alpha_2 = \alpha_3\varepsilon'$. Hence the left lower hand square is commutative.
(4) We consider the commutativity of the right lower hand square. Then

$$\varepsilon\beta_2(x\otimes h, y\otimes k) = \varepsilon(xh+yk) = 1+1$$
$$\beta_3\varepsilon^*(x\otimes h, y\otimes k) = \beta_3(x\otimes 1, y\otimes 1) = 1+1.$$

Thus we have $\varepsilon\beta_2 = \beta_3\varepsilon^*$. Hence the right lower hand square is commutative. Therefore we get the result by (1),(2),(3), and (4). □

As a consequence of the above theorem, we have the following corollary, which shows the evident relation between (3-3) and (3-5).

Corollary 3.6. *(3-3) is exact if and only if (3-5) is exact.*

Proof. The third column is given in (3-1). The first and second columns are given from (3-1) and by tensoring $\mathbb{Z}G\otimes_U -$ and $(\mathbb{Z}G\otimes_H -)\oplus(\mathbb{Z}G\otimes_K -)$ respectively. Then by Lemma 2.1 and Proposition 3.3 we get the result. □

Let G be the group defined by a given presentation $\wp = \langle\, \mathbf{x} : \mathbf{r}\,\rangle$ and let N be the normal closure of $\mathbf{r}$ in F, where F is the free group on $\mathbf{x}$. Then we have a short exact sequence of groups

$$(3-6) \qquad 1 \longrightarrow N \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1.$$

The abelianization N/N' of N can be regarded as a left $\mathbb{Z}G$-module via G-action induced by conjugation in F (if $U\in N$ and $W\in F$ then $(WN)(UN') = WUW^{-1}N'$). The G-module N/N' is called the *relation module* determined by the short exact sequence (3-6).

Next we consider the short exact sequences involving relation modules and augmentation ideals.

Lemma 3.7. *Let*

$$1 \longrightarrow N \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$$

be a short exact sequence of groups. Then

$$(3-10) \qquad 0 \longrightarrow N/N' \xrightarrow{\kappa} \mathbb{Z}G \otimes_F IF \xrightarrow{\nu} IG \longrightarrow 0$$

is an exact sequence of G-modules where $\kappa(UN') = 1_G \otimes (U-1)$ *and* $\nu(1_G \otimes (W-1)) = \pi(W) - 1 \ \ (U \in N,\ W \in F)$.

A proof of this lemma can be found in [9](Chapter VI, Theorem 6.3).

THEOREM 3.8. *The two short exact sequences (3-2) and (3-10) are isomorphic.*

Proof. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N/N' & \xrightarrow{\kappa} & \mathbb{Z}G \otimes_F IF & \xrightarrow{\nu} & IG & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \iota & & \\ 0 & \longrightarrow & IG \otimes_{\mathbb{Z}} IG & \xrightarrow{\gamma} & \mathbb{Z}G \otimes_{\mathbb{Z}} IG & \xrightarrow{\delta} & IG & \longrightarrow & 0 \end{array}$$

where α, β, γ, and δ are defined by

$\alpha : UN' \longmapsto 1_G \otimes (UN-1)\ (U \in N)$,
$\beta : 1_G \otimes (W-1) \longmapsto 1_G \otimes (WN-1)\ (W \in F)$,
$\gamma : (WN-1) \otimes (WN-1) \longmapsto (WN-1) \otimes (WN-1)\ (WN \in G)$,
$\delta : 1_G \otimes (WN-1) \longmapsto WN-1\ (WN \in G)$.

(1) We consider the commutativity of the left hand square. Then

$$\beta\kappa(UN') = \beta(1_G \otimes (U-1)) = 1_G \otimes (UN-1)$$
$$\gamma\alpha(UN') = \gamma(1_G \otimes (UN-1)) = 1_G \otimes (UN-1).$$

Thus we have $\beta\kappa = \gamma\alpha$. Hence the left hand square is commutative.
(2) We consider the commutativity of the right hand square. Then

$$\iota\nu(1_G \otimes (W-1)) = \iota(WN-1) = WN-1$$
$$\delta\beta(1_G \otimes (W-1)) = \delta(1_G \otimes (WN-1)) = WN-1.$$

Thus we have $\iota\nu = \delta\beta$. Hence the right hand square is commutative. Now we want to show that α is an isomorphism. We show that $ker\alpha = 0$. Let $UN' \in ker\alpha$. Then $0 = \gamma\alpha(UN') = \beta\kappa(UN')$. It is routine to show that β is an isomorphism. Since β is an isomorphism, $\kappa(UN') = 0$. Since κ is injective, it follows that $UN' = 0$. Secondly, we shall show that α is surjective. Let $(UN-1) \otimes (UN-1) \in IG \otimes_{\mathbb{Z}} IG$. Then $\gamma((UN-1) \otimes (UN-1)) \in \mathbb{Z}G \otimes_{\mathbb{Z}} IG$. Since β is an isomorphism, there exists $1_G \otimes (U-1) \in \mathbb{Z}G \otimes_F IF$ such that $\beta(1_G \otimes (U-1)) = \gamma((UN-1) \otimes (UN-1))$. Then $\iota\nu(1_G \otimes (U-1)) = \delta\beta(1_G \otimes (U-1)) = \delta\gamma((UN-1) \otimes (UN-1)) = 0$. Hence $\nu(1_G \otimes (U-1)) \in ker\iota$.

Since ι is an isomorphism, it follows that $\nu(1_G \otimes (U-1)) = 0$. Then $1_G \otimes (U-1) \in ker\nu = im\kappa$. Hence there exists $UN' \in N/N'$ such that $\kappa(UN') = 1_G \otimes (U-1)$. This implies that

$$\begin{aligned}
&\gamma(\alpha(UN') - (UN-1) \otimes (UN-1)) \\
&= \gamma\alpha(UN') - \gamma((UN-1) \otimes (UN-1)) \\
&= \beta\kappa(UN') - \gamma((UN-1) \otimes (UN-1)) \\
&= \beta(1_G \otimes (U-1)) - \gamma((UN-1) \otimes (UN-1)) \\
&= 0.
\end{aligned}$$

Then $\alpha(UN') - ((UN-1) \otimes (UN-1)) \in ker\gamma$. Since γ is injective, it follows that $\alpha(UN') - ((UN-1) \otimes (UN-1)) = 0$, i.e., $\alpha(UN') = (UN-1) \otimes (UN-1)$. Therefore α is surjective. Consequently, we obtain the result. □

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Department of Applied Mathematics
Pukyong National University
Pusan 608–737, Korea
E-mail: wooteks@hanmail.net