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A NOTE ON BITRANSFORMATION GROUPS

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Abstract. We study some dynamical properties in the context of bitransformation groups, and show that if (H, X, T) is a bitransformation group such that (H, X) is almost periodic and $(X/H, T)$ is pointwise almost periodic T_2 and $x \in X$, then $E_x = \{q \in E(H, X) \mid qx \in \overline{xT}\}$ is a compact T_2 topological group and $E_{qx} = E_x$ $(q \in E(H, X))$ when H is abelian, where $E(H, X)$ is the enveloping semigroup of the transformation group (H, X).

1. Introduction

A right transformation group (X, T) is a topological action $(x, t) \mapsto xt$ of the discrete group T on the compact Hausdorff space X. The enveloping semigroup $E(X, T)$ of the transformation group is a kind of compactification of the acting group and is itself a transformation group. The transformation group is *minimal* if every orbit is dense. Similarly a left transformation group (H, X) is a topological action $(h, x) \mapsto hx$ of the discrete group H on the compact Hausdorff space X.

A bitransformation group is a pair of transformation groups (H, X) and (X, T) with the same phase space X such that $h(xt) = (hx)t$ $(h \in H, x \in X, t \in T)$. The notation (H, X, T) will be used to signify that the pair (H, X), (X, T) constitute a bitransformation group, and $h(xt) = (hx)t$ will be denoted by hxt. In a bitransformation group (H, X, T), $(X/H, T)$ is compact but it need not be T_2. However if H is compact T_2, then $(X/H, T)$ is T_2 by [2; Lemma 4.9].

Let T be a topological group and A a subset of T. Then A is *syndetic* if there exists a compact subset K of T with $T = AK$. Let (X, T) be

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a transformation group and $x \in X$. Then x is an *almost periodic point* if $\{t \mid xt \in U\}$ is a syndetic subset of T for all neighborhoods U of x. A transformation group (X, T) is *pointwise almost periodic* iff each element of X is an almost periodic point [2].

A transformation group (X, T) is *weakly almost periodic* iff each element of $E(X, T)$ is continuous [4]. A transformation group (X, T) is *distal* iff $E(X, T)$ is group [1]. A transformation group (X, T) is (*uniformly*) *almost periodic* iff $E(X, T)$ is a group of homeomorphisms of X onto X [1].

The points x and y of a transformation group (X, T) are said to be *proximal* if there exists $p \in E(X, T)$ with $xp = yp$. Let (X, T) and (Y, T) be a transformation groups and let $\varphi : X \to Y$. Then φ is a *homomorphism* if φ is continuous and $\varphi(xt) = \varphi(x)t$ $(x \in X, t \in T)$. We say that a homomorphism $\varphi : X \to Y$ is *proximal* if whenever $x_1, x_2 \in \varphi^{-1}(y)$ then x_1 and x_2 are proximal [5].

In this paper we investigate some dynamical properties in the context of bitransformation groups.

2. Some characterizations of bitransformation groups

THEOREM 2.1. ([2]) *Let (H, X, T) be a bitransformation group such that X/H is T_2. Then $(X/H, T)$ is pointwise almost periodic iff (X, T) is pointwise almost periodic.*

Proof. This follows from Proposition 2.7 and Proposition 6.2 in [2]. □

THEOREM 2.2. ([2]) *Let (H, X, T) be a bitransformation group such that X/H is T_2. Then $(X/H, T)$ is distal iff (X, T) is distal.*

Proof. This follows from Corollary 5.7 and Proposition 6.6 in □

LEMMA 2.3. ([2]) *Let (H, X, T) be a bitransformation group such that X/H is T_2 and $(X/H, T)$ is minimal, and let $x \in X$. Then,*

1. *$\{\overline{hxT} \mid h \in H\}$ is a partition of X.*
2. *$H_x = \{h \in H \mid hx \in \overline{xT}\}$ is a closed subgroup of H with $H_{hx} = hH_xh^{-1}$ $(h \in H)$.*

3. *Let $\Re$ be the orbit closure relation on X. Then the map $\varphi : H \to X/\Re$ such that $\varphi(h) = \overline{hxT}$ induces a continuous bijective map $\overline{\varphi}$ of $H/H_x = \{gH_x \mid g \in H\}$ onto $X/\Re$. If in addition H is compact then $\Re$ is closed and $\overline{\varphi}$ is a homeomorphism.*

THEOREM 2.4. *Let (H, X, T) be a bitransformation group such that X/H is T_2. If $\psi : H \to X/H$ is a proximal homomorphism and $(X/H, T)$ is minimal, then (X, T) is minimal.*

Proof. Let $\psi : H \to X/H$ is a proximal homomorphism and $(X/H, T)$ is minimal. Then (X, T) contains a unique minimal set (see Lemma 1.1 of Chapter 2 in [5]). Since X/H is T_2 and $(X/H, T)$ is minimal, it follows from Lemma 2.3.1 that (X, T) is minimal. □

LEMMA 2.5. *Let (X, T) be a weakly almost periodic transformation group, with T abelian. Then $E(X, T)$ is also abelian.*

Proof. Let $p \in E(X, T)$, $t \in T$ and let $\{s_i\}$ be a net in T with $s_i \to p$. Since T is abelian and t is continuous, it follows that $ts_i = s_it$, $ts_i \to tp$, $s_it \to pt$, and $tp = pt$. Note that if (X, T) is weakly almost periodic, then each element of $E(X, T)$ is continuous. By continuity $E(X, T)$ is also abelian. □

THEOREM 2.6. *Let (X, T) be weakly almost periodic minimal, $x \in X$, and $H = \{p \in E(X, T) \mid xp = x\}$. Then the following are true.*

1. *(H, E, T) is a bitransformation group, where $E = E(X, T)$.*
2. *The flow $(E/H, T)$ is isomorphic with (X, T).*
3. *If $p \in E(X, T)$, then $\{\overline{qpT} \mid q \in H\}$ is a partition of $E(X, T)$ and $H_p = \{q \in H \mid qp \in \overline{pT}\}$ is a closed subgroup of H with $H_{qp} = qH_pq^{-1}$ $(q \in H)$.*
4. *Let $\Re$ be the orbit closure relation on $E(X, T)$. Then the map $\varphi : H \to E/\Re$ such that $\varphi(q) = \overline{qpT}$ induces a homeomorphism $\overline{\varphi}$ of $H/H_p = \{qH_p \mid q \in H\}$ onto $E/\Re$.*

Proof. Note that if (X, T) is weakly almost periodic and minimal, it is almost periodic by [1, Theorem 6 of chapter 4]. Thus E is a compact T_2 topological group [3] and H is a closed subgroup of E. This implies that H is a topological group and hence (H, E, T) is a bitransformation group. Since H is compact T_2, it follows that E/H is T_2. Since (X,T) is minimal, we have $\overline{xT} = X$. Therefore the map $\theta_x : p \longmapsto xp$ of E onto X induces an isomorphism of $(E/H,T)$ onto (X,T). The remainder of proof is obvious by Lemma 2.3. Note that if H is compact, then $\overline{\varphi}$ is a homeomorphism. □

Corollary 2.7. *Let (X,T) be a weakly almost periodic minimal transformation group, with T abelian. Then The flow (E,T) is isomorphic with (X,T).*

Proof. Let $x \in X$, $H = \{p \in E(X,T) \mid xp = x\}$, and $p \in H$. Since T is abelian, we have $(xt)p = xt$ for each $t \in T$. Let $y \in X$ and $\{s_i\}$ be a net in T with $xs_i \to y$. Then $yp = \lim(xs_i)p = \lim xs_i = y$, which implies that $H = \{e\}$. Therefore the isomorphism follows from Theorem 2.6.2. □

Corollary 2.8. *Let (X,T) be an almost periodic minimal transformation group, with T abelian. Then The flow (E,T) is isomorphic with (X,T).*

Theorem 2.9. *Let (H,X,T) be a bitransformation group such that (H,X) is almost periodic, and let $x \in X$. Then $E_x = \{q \in E(H,X) \mid qx \in \overline{xT}\}$ is a compact T_2 topological group.*

Proof. This follows from Lemma 2.3.2 and the fact that if (H,X) is almost periodic, then $E(H,X)$ is a compact T_2 topological group. □

Theorem 2.10. *Let (H,X,T) be a bitransformation group such that (H,X) is almost periodic and $(X/H,T)$ is pointwise almost periodic T_2, and let $x \in X$. Then the following are true.*

1. $q \in E_x$ iff $q\overline{xT} = \overline{xT}$.

2. $E_{qx} = qE_xq^{-1}$ $(q \in E(H,X))$.

3. If H is abelian, then $E_{qx} = E_x$ $(q \in E(H,X))$.

Proof. 1. Since (H,X,T) is a bitransformation group such that X/H is T_2 and $(X/H,T)$ is pointwise almost periodic, it follows that (X,T) is pointwise almost periodic by Theorem 2.1. Hence $\overline{xT}$ is a compact minimal subset of X. Then $q \in E_x$ iff $qx \in \overline{xT}$ iff $\overline{(qx)T} = \overline{xT}$. Since q is a homeomorphism of X onto X, we have $q\overline{xT} = \overline{qxT} = \overline{xT}$.

2. Let $q \in E(H,X)$. Since $E(H,X)$ is a group of homeomorphisms of X onto X, it follows that $p \in E_{qx} \Leftrightarrow \overline{pqxT} = \overline{qxT} \Leftrightarrow q^{-1}p\overline{qxT} = \overline{xT} \Leftrightarrow q^{-1}pq \in E_x \Leftrightarrow p \in qE_xq^{-1}$. Thus $E_{qx} = qE_xq^{-1}$.

3. Note that if H is abelian, then by continuity $E(H,X)$ is abelian. □

Now let G be a compact T_2 topological group, H a closed syndetic subgroup of G and $G/H = \{Hg \mid g \in G\}$. Then G/H is compact T_2 and the multiplication in G induces a transformation group structure on $(G/H, G)$. Since G is a compact group, $(G/H, G)$ is uniformly almost periodic. Since H is a subgroup of G, $(G/H, H)$ is uniformly almost periodic (see Proposition 1 of chapter 2 in [1]).

We denote η the neighborhood filter at the identity e of G, $P = P(G/H, G)$ the proximal relation on $(G/H, G)$ and $Q = Q(G/H, G)$ the regionally proximal relation on $(G/H, G)$.

Remark 2.11. ([2]) Since H is a closed syndetic subgroup of G, it follows that $P(G/H, G) = P(G/H, H)$ and $Q(G/H, G) = Q(G/H, H)$ (see Lemma 4.16 and Lemma 5.13 in [2]).

Theorem 2.12. *The following are equivalent:*
(a) $(G/H, G)$ *is almost periodic.*
(b) $(G/H, H)$ *is almost periodic.*
(c) $Q = \Delta$ *where* Δ *is the diagonal of* $G/H \times G/H$*.*
(d) $\bigcap_{U \in \eta} \overline{HUH} = H$*.*
(e) Given $U \in \eta$ *there exists* $V \in \eta$ *with* $VH \subset HU$*.*

Proof. Since H is a syndetic subgroup of G, it follows from [2, Proposition 4.17] that $(G/H, G)$ is almost periodic iff $(G/H, H)$ is almost periodic. The fact that $(G/H, G)$ is almost periodic iff Q is the diagonal of $G/H \times G/H$ follows from [2, Lemma 4.3]. It follows from [2, Corollary 6.8] that (a), (d),and (e) are pairwise equivalent. □

Theorem 2.13. *The following are equivalent:*
(a) $(G/H, G)$ *is distal.*
(b) $(G/H, H)$ *is distal.*
(c) $P = \Delta$ *where* Δ *is the diagonal of* $G/H \times G/H$*.*

(d) $\bigcap_{U\in\eta} HUH = H$.
(e) If $e \in \overline{HgH}$, *then* $g \in H$.

Proof. Since H is a syndetic subgroup of G, it follows from [2, Proposition 5.14] that $(G/H, G)$ is distal iff $(G/H, H)$ is distal. The fact that $(G/H, G)$ is distal iff P is the diagonal of $G/H \times G/H$ follows from [2, Lemma 5.12]. It follows from [1, Theorem 5.11] that (a), (d),and (e) are pairwise equivalent. □

Remark 2.14. ([2]) The following statements hold.
1. P is an equivalence relation on G/H iff $\bigcap_{U\in\eta} HUH$ is a group.
2. Q is an equivalence relation on G/H iff $\bigcap_{U\in\eta} \overline{HUH}$ is a group.

Remark 2.15. 1. It follows immediately from [2, Corollary 6.8] that $(G/H, G)$ is almost periodic when H is normal. Thus if G is abelian, then $(G/H, G)$ is always almost periodic.

2. If H is subnormal in G, then $(G/H, G)$ is distal (see Proposition 6.17 in [2]).

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