WEAK-star QUASI-SMOOTH \(\alpha\)-STRUCTURE
OF SMOOTH TOPOLOGICAL SPACES

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Abstract. In this paper we introduce the concepts of several types of weak-star quasi-smooth \(\alpha\)-compactness in terms of the concepts of weak smooth \(\alpha\)-closure and weak smooth \(\alpha\)-interior of a fuzzy set in smooth topological spaces and investigate some of their properties.

1. Introduction

Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang’s fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. Particularly, Gayyar, Kerre and Ramadan [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some of their properties. In [6] we introduced the concepts of smooth \(\alpha\)-closure and smooth \(\alpha\)-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined in [3] and also introduced several types of \(\alpha\)-compactness in smooth topological spaces and obtained some of their properties. In [7] we introduced the concepts of weak smooth \(\alpha\)-closure and weak smooth \(\alpha\)-interior of a fuzzy set in smooth topological spaces and investigated some of their properties.

In this paper we introduce the concepts of several types of weak-star quasi-smooth \(\alpha\)-compactness in terms of the concepts of weak smooth \(\alpha\)-closure and weak smooth \(\alpha\)-interior of a fuzzy set in smooth topological spaces and investigate some of their properties.

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2. Preliminaries

Let $X$ be a set and $I = [0, 1]$ be the unit interval of the real line. $I^{X}$ will denote the set of all fuzzy sets of $X$. $0_{X}$ and $1_{X}$ will denote the characteristic functions of $\phi$ and $X$, respectively.

A smooth topological space (s.t.s.) [8] is an ordered pair $(X, \tau)$, where $X$ is a non-empty set and $\tau : I^{X} \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_{X}) = \tau(1_{X}) = 1$;

(O2) $\forall A, B \in I^{X}, \; \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(O3) for any subfamily $\{A_{i} : i \in J\} \subseteq I^{X}, \; \tau(\bigcup_{i \in J} A_{i}) \geq \bigwedge_{i \in J} \tau(A_{i})$.

Then the mapping $\tau : I^{X} \to I$ is called a smooth topology on $X$. The number $\tau(A)$ is called the degree of openness of $A$.

A mapping $\tau^{*} : I^{X} \to I$ is called a smooth cotopology [8] if the following three conditions are satisfied:

(C1) $\tau^{*}(0_{X}) = \tau^{*}(1_{X}) = 1$;

(C2) $\forall A, B \in I^{X}, \; \tau^{*}(A \cup B) \geq \tau^{*}(A) \wedge \tau^{*}(B)$;

(C3) for every subfamily $\{A_{i} : i \in J\} \subseteq I^{X}, \; \tau^{*}(\bigcap_{i \in J} A_{i}) \geq \bigvee_{i \in J} \tau^{*}(A_{i})$.

If $\tau$ is a smooth topology on $X$, then the mapping $\tau^{*} : I^{X} \to I$, defined by $\tau^{*}(A) = \tau(A^{c})$ where $A^{c}$ denotes the complement of $A$, is a smooth cotopology on $X$. Conversely, if $\tau^{*}$ is a smooth cotopology on $X$, then the mapping $\tau : I^{X} \to I$, defined by $\tau(A) = \tau^{*}(A^{c})$, is a smooth topology on $X$ [8].

Demirci [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows:

Let $(X, \tau)$ be a s.t.s. and $A \in I^{X}$. Then the $\tau$-smooth closure (resp., $\tau$-smooth interior) of $A$, denoted by $\bar{A}$ (resp., $A^{o}$), is defined by $\bar{A} = \bigcap\{K \in I^{X} : \tau^{*}(K) > 0, A \subseteq K\}$ (resp., $A^{o} = \bigcup\{K \in I^{X} : \tau(K) > 0, K \subseteq A\}$). Demirci [4] defined the families $W(\tau) = \{A \in I^{X} : A = A^{c}\}$ and $W^{*}(\tau) = \{A \in I^{X} : A = A^{c}\}$, where $(X, \tau)$ is a s.t.s. Note that $A \in W(\tau)$ if and only if $A^{c} \in W^{*}(\tau)$.

Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces. A function $f : X \to Y$ is called smooth continuous with respect to $\tau$ and $\sigma$ [8] if $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^{Y}$. A function $f : X \to Y$ is called weakly smooth continuous with respect to $\tau$ and $\sigma$ [8] if $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^{Y}$. In this paper, a weakly smooth
continuous function with respect to $\tau$ and $\sigma$ is called a quasi-smooth continuous function with respect to $\tau$ and $\sigma$.

A function $f : X \to Y$ is smooth continuous with respect to $\tau$ and $\sigma$ if and only if $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to $\tau$ and $\sigma$ if and only if $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [8].

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to $\tau$ and $\sigma$ [8] if

$$\tau(A) \leq \sigma(f(A)) \quad \text{resp.,} \quad \tau^*(A) \leq \sigma^*(f(A))$$

for every $A \in I^X$.

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to $\tau$ and $\sigma$ [5] if

$$\sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$$

(resp., $\sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B))$)

for every $A, B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to $\tau$ and $\sigma$, then $\sigma^*(A) \geq \sigma^*(B)$ if and only if $\tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B)$ if and only if $\tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B))$) for every $A, B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to $\tau$ and $\sigma$ [5] if $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B))$) for every $A, B \in I^X$.

Let $(X, \tau)$ be a s.t.s., $\alpha \in [0, 1]$ and $A \in I^X$. The $\tau$-smooth $\alpha$-closure (resp., $\tau$-smooth $\alpha$-interior) of $A$, denoted by $\overline{A}_\alpha$ (resp., $A^c_\alpha$), is defined by $\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha \tau^*(A), A \subseteq K\}$ (resp., $A^c_\alpha = \cup\{K \in I^X : \tau(K) > \alpha \tau(A), K \subseteq A\}$) [6]. In [7] we defined the families $W_\alpha(\tau) = \{A \in I^X : A = A^c_\alpha\}$ and $W^*_\alpha(\tau) = \{A \in I^X : A = \overline{A}_\alpha\}$, where $(X, \tau)$ is a s.t.s. Note that $A \in W_\alpha(\tau) \Leftrightarrow A^c \in W^*_\alpha(\tau)$.

3. Types of weak* quasi-smooth $\alpha$-compactness

In this section, we introduce the concepts of several types of weak* quasi-smooth $\alpha$-compactness in smooth topological spaces and investigate some of their properties.
Definition 3.1[7]. Let \((X, \tau)\) be a s.t.s. \(\alpha \in [0, 1)\) and \(A \subseteq I^X\). The weak \(\tau\)-smooth \(\alpha\)-closure (resp., weak \(\tau\)-smooth \(\alpha\)-interior) of \(A\), denoted by \(wcl_\alpha(A)\) (resp., \(wint_\alpha(A)\)), is defined by \(wcl_\alpha(A) = \bigcap\{K \subseteq I^X : K \in W_\alpha(\tau), A \subseteq K\}\) (resp., \(wint_\alpha(A) = \bigcup\{K \subseteq I^X : K \in W_\alpha(\tau), K \subseteq A\}\)).

We define the families \(W_{wa}(\tau) = \{A \subseteq I^X : A = wint_\alpha(A)\}\) and \(W^{wa}_*(\tau) = \{A \subseteq I^X : A = wcl_\alpha(A)\}\), where \((X, \tau)\) is a s.t.s. and \(\alpha \in [0, 1)\). Then
\[
\begin{align*}
A \in W_{wa}(\tau) & \iff A^c \in W^{wa}_*(\tau), \\
A \in W_\alpha(\tau) & \Rightarrow A \subseteq W(\tau) \Rightarrow A \in W_{wa}(\tau), \\
A \in W^{wa}_*(\tau) & \Rightarrow A \subseteq W_\alpha(\tau) \Rightarrow A \in W^{wa}_*(\tau).
\end{align*}
\]

Definition 3.2[7]. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \(\alpha \in [0, 1)\). A function \(f : X \to Y\) is called weak \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\) if \(A \in W_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha(\tau)\) for every \(A \subseteq I^Y\).

Definition 3.3. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \(\alpha \in [0, 1)\). A function \(f : X \to Y\) is called weak* smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\) if \(A \in W_{wa}(\sigma) \Rightarrow f^{-1}(A) \in W_{wa}(\tau)\) for every \(A \subseteq I^Y\).

Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces. A function \(f : X \to Y\) is weak* smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\) if and only if \(A \in W^{wa}_*(\sigma) \Rightarrow f^{-1}(A) \in W^{wa}_*(\tau)\) for every \(A \subseteq I^Y\).

Definition 3.4. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \(\alpha \in [0, 1)\). A function \(f : X \to Y\) is called weak* smooth \(\alpha\)-open (resp., weak* smooth \(\alpha\)-closed) with respect to \(\tau\) and \(\sigma\) if \(A \in W_{wa}(\tau) \Rightarrow f(A) \in W_{wa}(\sigma)\) (resp., \(A \in W^{wa}_*(\tau) \Rightarrow f(A) \in W^{wa}_*(\sigma)\)) for every \(A \subseteq I^X\).

Theorem 3.5. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \(\alpha \in [0, 1)\). If a function \(f : X \to Y\) is weak smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\), then \(f : X \to Y\) is weak* smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\).
\textit{Proof.} Let \( f : X \to Y \) be a weak smooth \( \alpha \)-continuous function with respect to \( \tau \) and \( \sigma \). Then by Theorem 3.10[7] \( f^{-1}(\text{wint}_\alpha(A)) \subseteq \text{wint}_\alpha(f^{-1}(A)) \) for every \( A \in I^Y \). Let \( A \in W_{wa}(\sigma) \), i.e., \( A = \text{wint}_\alpha A \). Then \( f^{-1}(A) = f^{-1}(\text{wint}_\alpha A) \subseteq \text{wint}_\alpha(f^{-1}(A)) \). From the definition of weak smooth \( \alpha \)-interior we have \( \text{wint}_\alpha(f^{-1}(A)) \subseteq f^{-1}(A) \). Hence \( f^{-1}(A) = \text{wint}_\alpha(f^{-1}(A)) \), i.e., \( f^{-1}(A) \in W_{wa}(\tau) \). Therefore \( f : X \to Y \) is weak* smooth \( \alpha \)-continuous with respect to \( \tau \) and \( \sigma \).

\[ \square \]

**Definition 3.6.** Let \( \alpha \in [0, 1) \). A s.t.s. \((X, \tau)\) is called weak* quasi-smooth nearly \( \alpha \)-compact if for every family \( \{A_i : i \in J\} \) in \( W_{wa}(\tau) \) covering \( X \), there exists a finite subset \( J_0 \) of \( J \) such that \( \bigcup_{i \in J_0} \text{wint}_\alpha(\text{wcl}_\alpha(A_i)) = 1_X \).

**Definition 3.7.** Let \( \alpha \in [0, 1) \). A s.t.s. \((X, \tau)\) is called weak* quasi-smooth almost \( \alpha \)-compact if for every family \( \{A_i : i \in J\} \) in \( W_{wa}(\tau) \) covering \( X \), there exists a finite subset \( J_0 \) of \( J \) such that \( \bigcup_{i \in J_0} \text{wcl}_\alpha(A_i) = 1_X \).

Note that \((X, \tau)\) is weak* quasi-smooth almost \( \alpha \)-compact \( \Rightarrow (X, \tau)\) is weak* smooth almost compact \( \Rightarrow (X, \tau)\) is weak* smooth almost \( \alpha \)-compact.

**Theorem 3.8.** Let \((X, \tau)\) be a s.t.s. and let \( \alpha \in [0, 1) \). If \((X, \tau)\) is weak* smooth compact, then \((X, \tau)\) is weak* quasi-smooth nearly \( \alpha \)-compact.

**Proof.** Let \( \{A_i : i \in J\} \) be a family in \( W_{wa}(\tau) \) covering \( X \). Since \((X, \tau)\) is weak* smooth compact, there exists a finite subset \( J_0 \) of \( J \) such that \( \bigcup_{i \in J_0} A_i = 1_X \). Since \( A_i \in W_{wa}(\tau) \) for each \( i \in J \), \( A_i = \text{wint}_\alpha(A_i) \) for each \( i \in J \). From Theorem 3.3 and 3.4[7] we have \( \text{wint}_\alpha(A_i) \subseteq \text{wint}_\alpha(\text{wcl}_\alpha(A_i)) \) for each \( i \in J \). Thus \( 1_X = \bigcup_{i \in J_0} A_i = \bigcup_{i \in J_0} \text{wint}_\alpha(A_i) \subseteq \bigcup_{i \in J_0} \text{wint}_\alpha(\text{wcl}_\alpha(A_i)) \), i.e., \( \bigcup_{i \in J_0} \text{wint}_\alpha(\text{wcl}_\alpha(A_i)) = 1_X \). Hence \((X, \tau)\) is weak* quasi-smooth nearly \( \alpha \)-compact.

\[ \square \]

**Theorem 3.9.** Let \( \alpha \in [0, 1) \). Then a weak* quasi-smooth nearly \( \alpha \)-compact s.t.s. \((X, \tau)\) is weak* quasi-smooth almost \( \alpha \)-compact.
Proof. Let \((X, \tau)\) be a weak\(^*\) quasi-smooth nearly \(\alpha\)-compact s.t.s. Then for every family \(\{A_i : i \in J\} \) in \(W_{wa}(\tau)\) covering \(X\), there exists a finite subset \(J_0\) of \(J\) such that \(\cup_{i \in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X\). Since \(wint_\alpha(wcl_\alpha(A_i)) \subseteq wcl_\alpha(A_i)\) for each \(i \in J\) by Theorem 3.3[7], \(1_X = \cup_{i \in J_0} wint_\alpha(wcl_\alpha(A_i)) \subseteq \cup_{i \in J_0} wcl_\alpha(A_i)\). Thus \(\cup_{i \in J_0} wcl_\alpha(A_i) = 1_X\). Hence \((X, \tau)\) is weak\(^*\) quasi-smooth nearly \(\alpha\)-compact.

\[\square\]

Theorem 3.10. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces, \(\alpha \in [0,1)\) and \(f : X \to Y\) a surjective and weak smooth \(\alpha\)-continuous function with respect to \(\tau\) and \(\sigma\). If \((X, \tau)\) is weak\(^*\) quasi-smooth nearly \(\alpha\)-compact, then so is \((Y, \sigma)\).

Proof. Let \(\{A_i : i \in J\} \) be a family in \(W_{wa}(\sigma)\) covering \(Y\), i.e., \(\cup_{i \in J} A_i = 1_Y\). Then \(1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)\). Since \(f\) is weak smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\), \(f\) is weak\(^*\) smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\) by Theorem 3.5. Hence \(f^{-1}(A_i) \in W_{wa}(\tau)\) for each \(i \in J\). Since \((X, \tau)\) is weak\(^*\) quasi-smooth nearly \(\alpha\)-compact, there exists a finite subset \(J_0\) of \(J\) such that \(\cup_{i \in J_0} wcl_\alpha(f^{-1}(A_i)) = 1_X\). From the surjectivity of \(f\) we have \(1_Y = f(1_X) = f(\cup_{i \in J_0} wcl_\alpha(f^{-1}(A_i))) = \cup_{i \in J_0} f(wcl_\alpha(f^{-1}(A_i)))\). Since \(f : X \to Y\) is weak smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\), from Theorem 3.10[7] we have \(wcl_\alpha(f^{-1}(A)) \subseteq f^{-1}(wcl_\alpha(A))\) for every \(A \in I^Y\). Hence \(1_Y = \cup_{i \in J_0} f(wcl_\alpha(f^{-1}(A_i))) \subseteq \cup_{i \in J_0} f(f^{-1}(wcl_\alpha(A_i))) = \cup_{i \in J_0} wcl_\alpha(A_i)\), i.e., \(\cup_{i \in J_0} wcl_\alpha(A_i) = 1_Y\). Thus \((Y, \sigma)\) is weak\(^*\) quasi-smooth nearly \(\alpha\)-compact.

\[\square\]

Theorem 3.11. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces, \(\alpha \in [0,1)\) and \(f : X \to Y\) a surjective, weak smooth \(\alpha\)-continuous and weak smooth \(\alpha\)-open function with respect to \(\tau\) and \(\sigma\). If \((X, \tau)\) is weak\(^*\) quasi-smooth nearly \(\alpha\)-compact, then so is \((Y, \sigma)\).

Proof. Let \(\{A_i : i \in J\} \) be a family in \(W_{wa}(\sigma)\) covering \(Y\), i.e., \(\cup_{i \in J} A_i = 1_Y\). Then \(1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)\). Since \(f\) is weak smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\), \(f\) is weak\(^*\) smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\) by Theorem 3.5. Hence \(f^{-1}(A_i) \in W_{wa}(\tau)\) for each \(i \in J\). Since \((X, \tau)\) is weak\(^*\) quasi-smooth nearly
\(\alpha\)-compact, there exists a finite subset \(J_0\) of \(J\) such that 
\[\bigcup_{i \in J_0} \text{wint}_\alpha (\text{wcl}_\alpha (f^{-1}(A_i))) = 1_X.\]
From the surjectivity of \(f\) we have
\[1_Y = f(1_X) = f\left(\bigcup_{i \in J_0} \text{wint}_\alpha (\text{wcl}_\alpha (f^{-1}(A_i)))\right)\]
\[= \bigcup_{i \in J_0} f(\text{wint}_\alpha (\text{wcl}_\alpha (f^{-1}(A_i)))).\]

Since \(f : X \rightarrow Y\) is weak smooth \(\alpha\)-open with respect to \(\tau\) and \(\sigma\), from Theorem 3.12[7] we have
\[f(\text{wint}_\alpha (\text{wcl}_\alpha (f^{-1}(A_i)))) \subseteq \text{wint}_\alpha (f(\text{wcl}_\alpha (f^{-1}(A_i))))\]
for each \(i \in J\). Since \(f : X \rightarrow Y\) is weak smooth \(\alpha\)-continuous with respect to \(\tau\) and \(\sigma\), from Theorem 3.10[7] we have \(\text{wcl}_\alpha (f^{-1}(A_i)) \subseteq f^{-1}(\text{wcl}_\alpha (A_i))\) for each \(i \in J\). Hence we have
\[1_Y = \bigcup_{i \in J_0} f(\text{wint}_\alpha (\text{wcl}_\alpha (f^{-1}(A_i))))\]
\[\subseteq \bigcup_{i \in J_0} \text{wint}_\alpha (f(\text{wcl}_\alpha (f^{-1}(A_i))))\]
\[\subseteq \bigcup_{i \in J_0} \text{wint}_\alpha (f(f^{-1}(\text{wcl}_\alpha (A_i))))\]
\[= \bigcup_{i \in J_0} \text{wint}_\alpha (\text{wcl}_\alpha (A_i)).\]

Thus \(\bigcup_{i \in J_0} \text{wint}_\alpha (\text{wcl}_\alpha (A_i)) = 1_Y\). Hence \((Y, \sigma)\) is weak\(^*\) quasi-smooth nearly \(\alpha\)-compact.

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