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G-FUZZY CONGRUENCES GENERATED BY COMPATIBLE FUZZY RELATIONS

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Abstract. We define a G-fuzzy congruence, which is a generalized fuzzy congruence, and characterize the G-fuzzy congruence generated by a left and right compatible fuzzy relation on a semigroup.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh [9]. Subsequently, Goguen [2] and Sanchez [7] studied fuzzy relations in various contexts. In [5] Nemitz discussed fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali [4] developed some properties of fuzzy equivalence relations and certain lattice theoretic properties of fuzzy equivalence relations. Samhan [6] discussed the fuzzy congruence generated by a fuzzy relation on a semigroup and studied the lattice of fuzzy congruences on a semigroup. Gupta et al. [3] proposed a generalized definition of a fuzzy equivalence relation on a set, which we call G-fuzzy equivalence relation in this paper, and developed some properties of that relation. In [8] Tan developed some properties of fuzzy congruences on a regular semigroup. Chon [1] characterized the G-fuzzy congruence generated by a fuzzy relation on a semigroup and gave some lattice theoretic properties of G-fuzzy congruences on semigroups. The present work has been started as a continuation of these studies.

In section 2 we define a G-fuzzy congruence and review some basic definitions and properties of fuzzy relations and G-fuzzy congruences. In section 3 we find the G-fuzzy congruence generated by a

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left and right compatible fuzzy relation μ on a semigroup S such that $\sup_{x \neq y \in S} \mu(x, y) > 0$ for some $x \neq y \in S$, find the G-fuzzy congruence generated by a left and right compatible fuzzy relation μ on a semigroup S such that $\mu(x, y) = 0$ for all $x \neq y \in S$ and $\mu(z, z) > 0$ for all $z \in S$, and show that there does not exist the G-fuzzy congruence generated by a left and right compatible fuzzy relation μ on a semigroup S such that $\mu(x, y) = 0$ for all $x \neq y \in S$ and $\mu(z, z) = 0$ for some $z \in S$.

2. Preliminaries

We recall some basic definitions and properties of fuzzy relations and G-fuzzy congruences which will be used in the next section.

DEFINITION 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

The standard definition of a fuzzy reflexive relation μ in a set X demands $\mu(x, x) = 1$. Gupta et al. ([3]) weakened this definition as follows.

DEFINITION 2.2. A *fuzzy relation* μ in a set X is a fuzzy subset of $X \times X$. μ is *G-reflexive* in X if $\mu(x, x) > 0$ and $\mu(x, y) \leq \inf_{t \in X} \mu(t, t)$ for all $x, y \in X$ such that $x \neq y$. μ is *symmetric* in X if $\mu(x, y) = \mu(y, x)$ for all x, y in X. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x, y) = \sup_{z \in X} \min(\lambda(x, z), \mu(z, y)).$$

A fuzzy relation μ in X is *transitive* in X if $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called *G-fuzzy equivalence relation* if μ is G-reflexive, symmetric, and transitive.

DEFINITION 2.3. Let μ be a fuzzy relation in a set X. μ is called *fuzzy left (right) compatible* if $\mu(x, y) \leq \mu(zx, zy)$ $(\mu(x, y) \leq \mu(xz, yz))$ for all $x, y, z \in X$. A G-fuzzy equivalence relation on X is called a *G-fuzzy left congruence (right congruence)* if it is fuzzy left compatible (right compatible). A G-fuzzy equivalence relation on X is a *G-fuzzy congruence* if it is a G-fuzzy left and right congruence.

DEFINITION 2.4. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x, y) = \mu(y, x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations μ and ν.

PROPOSITION 2.5. *Let μ be a fuzzy relation on a set X. Then $\cup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on X containing μ, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.3 of [6]. □

PROPOSITION 2.6. *Let μ be a fuzzy relation on a set X. If μ is symmetric, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.4 of [6]. □

PROPOSITION 2.7. *If μ is a fuzzy relation on a semigroup S that is fuzzy left and right compatible, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 3.6 of [6]. □

3. G-fuzzy congruences generated by fuzzy relations

In this section we characterize the G-fuzzy congruence generated by a left and right compatible fuzzy relation on a semigroup.

PROPOSITION 3.1. *Let μ be a fuzzy relation on a set S. If μ is G-reflexive, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. Clearly $\mu^1 = \mu$ is G-reflexive. Suppose μ^k is G-reflexive.

$$\begin{aligned}\mu^{k+1}(x, x) = (\mu^k \circ \mu)(x, x) &= \sup_{z \in S} \min[\mu^k(x, z), \mu(z, x)] \\ &\geq \min[\mu^k(x, x), \mu(x, x)] > 0\end{aligned}$$

for all $x \in S$. Let $x, y \in S$ with $x \neq y$. Then

$$\begin{aligned}\inf_{t\in S} \mu^{k+1}(t,t) &= \inf_{t\in S} (\mu^k \circ \mu)(t,t) \\ &= \inf_{t\in S} \sup_{z\in S} \min[\mu^k(t,z), \mu(z,t)] \\ &\geq \inf_{t\in S} \min[\mu^k(t,t), \mu(t,t)] \\ &\geq \min\,[\inf_{t\in S} \mu^k(t,t), \inf_{t\in S} \mu(t,t)] \geq \min[\mu^k(x,z), \mu(z,y)]\end{aligned}$$

for all $z \in S$ such that $z \neq x$ and $z \neq y$. That is,

$$\inf_{t\in S} \mu^{k+1}(t,t) \geq \sup_{z\in S-\{x,y\}} \min[\mu^k(x,z), \mu(z,y)].$$

Clearly

$$\inf_{t\in S} \mu(t,t) \geq \min\,[\mu^k(x,x), \mu(x,y)]$$

and

$$\inf_{t\in S} \mu^k(t,t) \geq \min\,[\mu^k(x,y), \mu(y,y)].$$

Since $\mu^{k+1}(t,t) \geq \mu^k(t,t) \geq \mu(t,t)$ for $k \geq 1$,

$$\inf_{t\in S} \mu^{k+1}(t,t) \geq \min\,[\mu^k(x,x), \mu(x,y)]$$

and

$$\inf_{t\in S} \mu^{k+1}(t,t) \geq \min\,[\mu^k(x,y), \mu(y,y)].$$

Thus

$$\begin{aligned}\inf_{t\in S} \mu^{k+1}(t,t) \geq \max\,[&\sup_{z\in S-\{x,y\}} \min(\mu^k(x,z), \mu(z,y)), \\ &\min\,(\mu^k(x,x), \mu(x,y)),\ \min\,(\mu^k(x,y), \mu(y,y))] \\ = &\sup_{z\in S} \min[\mu^k(x,z), \mu(z,y)] \\ = &(\mu^k \circ \mu)(x,y) = \mu^{k+1}(x,y).\end{aligned}$$

That is, μ^{k+1} is G-reflexive. By the mathematical induction, μ^n is G-reflexive for $n = 1, 2, \ldots$. Thus $\inf_{t\in S} [\cup_{n=1}^{\infty} \mu^n](t,t) = \inf_{t\in S} \sup[\mu(t,t), (\mu \circ \mu)(t,t), \ldots] \geq \sup\,[\inf_{t\in S} \mu(t,t), \inf_{t\in S} (\mu \circ \mu)(t,t), \ldots] \geq \sup[\mu(x,y), (\mu \circ \mu)(x,y), \ldots] = [\cup_{n=1}^{\infty} \mu^n](x,y)$. Clearly $[\cup_{n=1}^{\infty} \mu^n](x,x) > 0$. Hence $\cup_{n=1}^{\infty} \mu^n$ is G-reflexive. □

Theorem 3.2. *Let μ be a fuzzy relation on a semigroup S such that μ is fuzzy left and right compatible.*

(1) *If $\mu(x,y) > 0$ for some $x \neq y \in S$, then the G-fuzzy congruence generated by μ is $\cup_{n=1}^{\infty} [\mu \cup \mu^{-1} \cup \theta]^n$, where θ is a fuzzy relation on S such that $\theta(z,z) = \sup\limits_{x \neq y \in S} \mu(x,y)$ for all $z \in S$ and $\theta(x,y) = \theta(y,x) \leq \min\, [\mu(x,y), \mu(y,x)]$ for all $x, y \in S$ with $x \neq y$.*

(2) *If $\mu(x,y) = 0$ for all $x \neq y \in S$ and $\mu(z,z) > 0$ for all $z \in S$, then the G-fuzzy congruence generated by μ is $\cup_{n=1}^{\infty} \mu^n$.*

(3) *If $\mu(x,y) = 0$ for all $x \neq y \in S$ and $\mu(z,z) = 0$ for some $z \in S$, then there does not exist the G-fuzzy congruence generated by μ.*

Proof. (1) Let $\mu_1 = \mu \cup \mu^{-1} \cup \theta$. Since $\theta(z,z) > 0$, $\mu_1(z,z) > 0$ for all $z \in S$. Let $x, y \in S$ with $x \neq y$. Then $\theta(x,y) \leq \mu(x,y) \leq \sup\limits_{x \neq y \in S} \mu(x,y) = \theta(t,t)$ for all $t \in S$. Thus

$$\begin{aligned}\inf_{t\in S} \mu_1(t,t) &\geq \inf_{t \in S} \theta(t,t) \\ &\geq \max[\mu(x,y),\ \mu^{-1}(x,y),\ \theta(x,y)] = \mu_1(x,y).\end{aligned}$$

That is, μ_1 is G-reflexive. By Proposition 3.1, $\cup_{n=1}^{\infty} \mu_1^n$ is G-reflexive. Since $\theta(x,y) = \theta(y,x)$, $\theta = \theta^{-1}$. Thus

$$\begin{aligned}\mu_1(x,y) &= \max\, [\mu(x,y), \mu^{-1}(x,y), \theta(x,y)] \\ &= \max\, [\mu^{-1}(y,x), \mu(y,x), \theta^{-1}(x,y)] \\ &= \max[\mu^{-1}(y,x), \mu(y,x), \theta(y,x)] \\ &= \mu_1(y,x).\end{aligned}$$

That is, μ_1 is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty} \mu_1^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty} \mu_1^n$ is transitive. Hence $\cup_{n=1}^{\infty} \mu_1^n$ is a G-fuzzy equivalence relation containing μ. Since $\theta(x,y) \leq \mu(x,y) \leq \mu(zx,zy)$,

$$\begin{aligned}\mu_1(x,y) &= \max\, [\mu(x,y), \mu^{-1}(x,y), \theta(x,y)] \\ &= \max\, [\mu(x,y), \mu(y,x), \theta(x,y)] = \max\, [\mu(x,y), \mu(y,x)] \\ &\leq \max\, [\mu(zx,zy), \mu(zy,zx)] \\ &\leq \max\, [\mu(zx,zy), \mu(zy,zx), \theta(zx,zy)] \\ &= \max\, [\mu(zx,zy), \mu^{-1}(zx,zy), \theta(zx,zy)] = \mu_1(zx,zy)\end{aligned}$$

for all $x, y, z \in S$ such that $x \neq y$. Since $\theta(x,x) = \theta(zx,zx)$ for all $x, z \in S$, $\mu_1(x,x) = \max\,[\mu(x,x), \mu^{-1}(x,x), \theta(x,x)] \leq \max\,[\mu(zx,zx), \theta(zx,zx)] = \max\,[\mu(zx,zx), \mu^{-1}(zx,zx), \theta(zx,zx)] = \mu_1(zx,zx)$ for all $x, z \in S$. Thus μ_1 is fuzzy left compatible. Similarly we may show μ_1 is fuzzy right compatible. By Proposition 2.7, $\cup_{n=1}^{\infty}\ \mu_1^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty}\ \mu_1^n$ is a G-fuzzy congruence containing μ. Let ν be a G-fuzzy congruence containing μ. Then $\mu(x,y) \leq \nu(x,y)$, $\mu^{-1}(x,y) = \mu(y,x) \leq \nu(y,x) = \nu(x,y)$, and $\theta(x,y) \leq \mu(x,y) \leq \nu(x,y)$. Thus $\mu_1(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$. Since $\nu(a,a) \geq \nu(x,y) \geq \mu(x,y)$ for all $a, x, y \in S$ such that $x \neq y$, $\theta(a,a) = \sup\limits_{x \neq y \in S} \mu(x,y) \leq \nu(a,a)$ for all $a \in S$. Since $\nu(a,a) \geq \mu(a,a) = \mu^{-1}(a,a)$ and $\nu(a,a) \geq \theta(a,a)$ for all $a \in S$, $\max\,[\mu(a,a), \mu^{-1}(a,a), \theta(a,a)] \leq \nu(a,a)$ for all $a \in S$. Thus $\mu_1 \subseteq \nu$. Suppose $\mu_1^k \subseteq \nu$. Then $\mu_1^{k+1}(b,c) = (\mu_1^k \circ \mu_1)(b,c) = \sup\limits_{d \in S} \min[\mu_1^k(b,d), \mu_1(d,c)] \leq \sup\limits_{d \in S} \min\,[\nu(b,d), \nu(d,c)] = (\nu \circ \nu)(b,c)$ for all $b, c \in S$. That is, $\mu_1^{k+1} \subseteq (\nu \circ \nu)$. Since ν is transitive, $\mu_1^{k+1} \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for every natural number n. Thus $\cup_{n=1}^{\infty}\ [\mu \cup \mu^{-1} \cup \theta]^n = \cup_{n=1}^{\infty}\ {\mu_1}^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu$.

(2) Let $x, y \in S$ with $x \neq y$. Since $\mu(x,y) = 0$, $\inf\limits_{t \in S} \mu(t,t) \geq \mu(x,y)$. Thus μ is G-reflexive. Since $\mu(x,y) = 0$, μ is symmetric. By Proposition 2.5, Proposition 2.6, and Proposition 3.1, $\cup_{n=1}^{\infty}\ \mu^n$ is a G-fuzzy equivalence relation containing μ. Since μ is fuzzy left and right compatible from the hypothesis, $\cup_{n=1}^{\infty}\ \mu^n$ is a G-fuzzy congruence containing μ by Proposition 2.7. Let ν be a G-fuzzy congruence containing μ. By the mathematical induction as shown in Theorem 3.2 (1), we may show that $\mu^n \subseteq \nu$ for every natural number n. Hence $\cup_{n=1}^{\infty}\ \mu^n = \mu \cup (\mu \circ \mu) \cup (\mu \circ \mu \circ \mu) \cdots \subseteq \nu$.

(3) Suppose ξ is the G-fuzzy congruence generated by μ. Then $\xi(z,z) > 0$ for every $z \in S$. Let θ be a fuzzy relation such that $\theta(a,b) = \frac{\xi(a,b)}{2}$ for all $a, b \in S$. Then $\theta(z,z) > 0$ for all $z \in S$. Let $x, y \in S$ with $x \neq y$. Since ξ is G-reflexive, $\inf\limits_{t \in S} \xi(t,t) \geq \xi(x,y)$. Since $\theta(a,b) = \frac{\xi(a,b)}{2}$ for all $a, b \in S$, $\inf\limits_{t \in S} \theta(t,t) \geq \theta(x,y)$. Since $\mu(x,y) = 0$, $\inf\limits_{t \in S}\ (\mu \cup \theta)(t,t) \geq \inf\limits_{t \in S} \theta(t,t) \geq (\mu \cup \theta)(x,y)$. That is, $\mu \cup \theta$ is G-reflexive. Since ξ is symmetric, θ is symmetric. Since θ is symmetric

and $\mu(x,y)=0$, $\mu\cup\theta=(\mu\cup\theta)^{-1}$. That is, $\mu\cup\theta$ is symmetric. By Proposition 2.5, Proposition 2.6, and Proposition 3.1, $\cup_{n=1}^{\infty}\ (\mu\cup\theta)^n$ is a G-fuzzy equivalence relation containing μ. Since $\theta(a,b)=\frac{\xi(a,b)}{2}$ for all $a,b\in S$ and ξ is fuzzy left and right compatible, θ is fuzzy left and right compatible. Since μ is fuzzy left and right compatible, $\mu\cup\theta$ is fuzzy left and right compatible. By Proposition 2.7, $\cup_{n=1}^{\infty}\ (\mu\cup\theta)^n$ is a G-fuzzy congruence containing μ. Since $\theta(a,b)=\frac{\xi(a,b)}{2}\leq\xi(a,b)$ and $\mu(a,b)\leq\xi(a,b)$ for all $a,b\in S$, $\mu\cup\theta\subseteq\xi$. Let $\mu_1=\mu\cup\theta$. Then $\mu_1\subseteq\xi$. By the mathematical induction as shown in Theorem 3.2 (1), we may show that $\mu_1^n\subseteq\xi$ for every natural number n. Hence $\cup_{n=1}^{\infty}\ [\mu\cup\theta]^n=\cup_{n=1}^{\infty}\ {\mu_1}^n\subseteq\xi$. Let $v\neq w\in S$. Then $\mu_1(v,w)=(\mu\cup\theta)(v,w)=\theta(v,w)\leq\inf\limits_{t\in S}\theta(t,t)\leq\mu_1(z,z)$ for every $z\in S$. Suppose $\mu_1^k(v,w)\leq\mu_1(z,z)$ for every $z\in S$. Then

$$
\begin{aligned}
\mu_1^{k+1}(v,w) &= \sup_{s\in S}\ \min\ [\mu_1^k(v,s),\ \mu_1(s,w)]\\
&= \max\ [\sup_{s\in S-\{v,w\}}\ \min(\mu_1^k(v,s),\ \mu_1(s,w)),\\
&\qquad \min\ (\mu_1^k(v,v),\mu_1(v,w)),\ \min\ (\mu_1^k(v,w),\mu_1(w,w))]\\
&\leq \max\ [\mu_1(z,z),\ \mu_1(z,z),\ \mu_1^k(v,w)]=\mu_1(z,z).
\end{aligned}
$$

By the mathematical induction, $\mu_1^n(v,w)\leq\mu_1(z,z)$ for every natural number n. Clearly $\mu_1^k(z,z)=\mu_1(z,z)$ for $k=1$. Suppose $\mu_1^k(z,z)=\mu_1(z,z)$. Since $\mu_1^k(z,s)\leq\mu_1(z,z)$ for $s\neq z\in S$, $\mu_1^{k+1}(z,z)=\sup\limits_{s\in S}\ \min\ [\mu_1^k(z,s),\mu_1(s,z)]=\max\ [\sup\limits_{s\in S-\{z\}}\ \min(\mu_1^k(z,s),\ \mu_1(s,z)),$ $\min\ (\mu_1^k(z,z),\mu_1(z,z))]=\mu_1(z,z)$. By the mathematical induction, $\mu_1^n(z,z)=\mu_1(z,z)$ for every natural number n and every $z\in S$. Let p be in S with $\mu(p,p)=0$. Then $\mu_1(p,p)=\theta(p,p)=\frac{\xi(p,p)}{2}<\xi(p,p)$. Since $\mu_1^n(z,z)=\mu_1(z,z)$ for every natural number n and every $z\in S$, $[\cup_{n=1}^{\infty}\ (\mu\cup\theta)^n](p,p)=[\cup_{n=1}^{\infty}\ {\mu_1}^n](p,p)=\mu_1(p,p)<\xi(p,p)$ for some $p\in S$ such that $\mu(p,p)=0$. Hence $\cup_{n=1}^{\infty}\ (\mu\cup\theta)^n$, which is a G-fuzzy congruence containing μ, is contained in ξ. This contradicts that ξ is the G-fuzzy congruence generated by μ. □

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