

CRITICAL POINTS AND MULTIPLE SOLUTIONS OF A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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ABSTRACT. We consider a semilinear elliptic boundary value problem with Dirichlet boundary condition $Au + bu^+ - au^- = t_1\phi_1 + t_2\phi_2$ in Ω and ϕ_n is the eigenfunction corresponding to λ_n ($n = 1, 2, \dots$). We have a concern with the multiplicity of solutions of the equation when $\lambda_1 < a < \lambda_2 < b < \lambda_3$.

1. Introduction

Let Ω be a bounded set in $\mathbf{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$ and let A denote the elliptic operator

$$(1.1) \quad A = \sum_{1 \leq i, j \leq n} a_{i,j}(x) D_i D_j,$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$.

We consider a semilinear elliptic equation with Dirichlet boundary condition

$$(1.2) \quad \begin{aligned} Au + bu^+ - au^- &= h(x) \quad \text{in } \Omega. \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here A is a second order elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated as often as multiplicity. We denote ϕ_n to be the eigenfunction corresponding to λ_n ($n = 1, 2, \dots$), and ϕ_1 is the eigenfunction such that

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$\phi_1 > 0$ in Ω and the set $\{\phi_n | n = 1, 2, 3 \dots\}$ is an orthonormal set in H , where H is a Hilbert space with inner product

$$(u, v) = \int_{\Omega} uv, \quad u, v \in L^2(\Omega).$$

We suppose that $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Under these assumptions, we have a concern with the multiplicity of solutions of (1.2) when h is generated by two eigenfunctions ϕ_1 and ϕ_2 . Then equation (1.2) is equivalent to

$$(1.3) \quad Au + bu^+ - au^- = h \quad \text{in } H,$$

where $h = t_1\phi_1 + t_2\phi_2$ ($t_1, t_2 \in \mathbf{R}$). Hence we will study the equation (1.3). To study equation (1.3), We use the contraction mapping principle to reduce the problem from an infinite dimensional space in H to a finite dimensional one.

Let V be the two dimensional subspace of H spanned by $\{\phi_1, \phi_2\}$ and W be the orthogonal complement of V in H . Let P be an orthogonal projection H onto V . Then every element $u \in H$ is expressed as

$$u = v + w,$$

where $v = Pu, w = (I - P)u$. Hence equation (1.3) is equivalent to a system

$$(1.4) \quad Aw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0$$

$$(1.5) \quad Av + P(b(v + w)^+ - a(v + w)^-) = t_1\phi_1 + t_2\phi_2.$$

Here we look on (1.4) and (1.5) as a system of two equation in the two unknowns v and w . We can see that for fixed $v \in V$, (1.4) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the L^2 -norm) in terms of v .

The study of the multiplicity of solution of (1.3) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$(1.6) \quad Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = t_1\phi_1 + t_2\phi_2$$

defined on the two dimensional subspace V spanned by $\{\phi_1, \phi_2\}$.

While one feels intuitively that (1.6) ought to be easier to solve than (1.3), there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some special v 's.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For example, let us take $v \geq 0$ and $\theta(v) = 0$. Then equation (1.4) reduces to

$$A0 + (I - P)(bv^+ - av^-) = 0,$$

which is satisfied because $v^+ = v, v^- = 0$ and $(I - P)v = 0$, since $v \in V$. Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and ϕ_1 is a positive eigenfunction, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \geq 0, |c_2| \leq qc_1\}$$

for some $q > 0$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \leq 0, |c_2| \leq q|c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

Thus, even if we do not know $\theta(v)$ for all $v \in V$. we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$.

2. The existence of solutions and source terms

Now we define a map $\Pi : V \rightarrow V$ given by

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V.$$

Then, we can obtain that the following theorem.

THEOREM 2.1. $\Pi(cv) = c\Pi(v)$ for $c \geq 0$.

We investigate the image of the cones C_1, C_3 under Π . First, we consider the image of cone C_1 . If $v = c_1\phi_1 + c_2\phi_2 \geq 0$, we have

$$\begin{aligned} \Pi(v) &= Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\ &= -c_1\lambda_1\phi_1 - c_2\lambda_2\phi_2 + b(c_1\phi_1 + c_2\phi_2) \\ &= c_1(b - \lambda_1)\phi_1 + c_2(b - \lambda_2)\phi_2. \end{aligned}$$

Thus the image of the rays $c_1\phi_1 \pm qc_1\phi_2 (c_1 \geq 0)$ can explicitly calculated and they are

$$(2.1) \quad c_1(b - \lambda_1)\phi_1 \pm qc_1(b - \lambda_2)\phi_2 \quad (c_1 \geq 0).$$

Therefore If $\lambda_1 < a < \lambda_2 < b < \lambda_3$, then Π maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_1 + d_2\phi_2 \mid d_1 \geq 0, |d_2| \leq q \left(\frac{b - \lambda_2}{b - \lambda_1} \right) d_1 \right\}.$$

Second, similarly, the image of the rays $-c_1\phi_1 \pm qc_1\phi_2 (c_1 \geq 0)$ are

$$(2.2) \quad c_1(\lambda_1 - a)\phi_1 \pm qc_1(\lambda_2 - a)\phi_2 \quad (c_1 \geq 0).$$

Therefore, if $\lambda_1 < a < \lambda_2 < b < \lambda_3$, then Π maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_1 + d_2\phi_2 \mid d_1 \leq 0, |d_2| \leq q \left(\frac{\lambda_2 - a}{\lambda_1 - a} \right) d_1 \right\}.$$

Now we set

$$C_2 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_2 \geq 0, c_2 \geq q|c_1|\},$$

$$C_4 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_2 \leq 0, |c_2| \geq q|c_1|\},$$

Then the union of C_1, C_2 , and C_3, C_4 are the space V .

We remember the map $\Pi : V \rightarrow V$ given by

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V.$$

Let $R_i (1 \leq i \leq 4)$ be the image of $C_i (1 \leq i \leq 4)$ under Π .

THEOREM 2.2. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$.*

(a) If h belongs to R_1 , then equation (1.2) has a positive solution and no negative solution. If h belongs to R_3 , then equation (1.2) has a negative solution.

(b) For $i = 1, 3$, the image of Π_i is R_i and $\Pi_i : C_i \rightarrow R_i$ is bijective.

Proof. (a) From (2.1) and (2.2), if h belongs to R_1 , the equation $\Pi(v) = t_1\phi_1 + t_2\phi_2$ has a positive solution in the cone C_1 , namely $\frac{t_1}{b-\lambda_1}\phi_1 + \frac{t_2}{b-\lambda_2}\phi_2$, and if h belongs to R_3 , the equation $\Pi(v) = t_1\phi_1 + t_2\phi_2$ has a negative solution in C_3 , namely $-\frac{t_1}{\lambda_1-a}\phi_1 - \frac{t_2}{\lambda_2-a}\phi_2$.

(b) We consider the restriction Π_1 . By (2.1), the restriction Π_1 maps C_1 onto R_1 . Let l_1 be the segment defined by

$$l_1 = \left\{ \phi_1 + d_2\phi_2 \mid |d_2| \leq q \left(\frac{b-\lambda_2}{b-\lambda_1} \right) \right\}.$$

Then the inverse image $\Pi_1^{-1}(l_1)$ is a segment

$$\mathcal{L}_1 = \left\{ \frac{1}{b-\lambda_1}(\phi_1 + c_2\phi_2) \mid |c_2| \leq q \right\}.$$

It follows from Theorem 2.1 that $\Pi_1 : C_1 \rightarrow R_1$ is bijective. Similarly, $\Pi_3 : C_3 \rightarrow R_3$ is also a bijection. $\square$

We set

$$C_2 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_2 \geq 0, c_2 \geq q|c_1|\},$$

$$C_4 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_2 \leq 0, |c_2| \geq q|c_1|\}.$$

Then the union of C_1, C_2 , and C_3, C_4 is the space V . Theorem 2.1 means that the images $\Pi(C_2)$ and $\Pi(C_4)$ are the cones in the plane V . Before we investigate the images $\Pi(C_2)$ and $\Pi(C_4)$, we set

$$R_2^* = \left\{ d_1\phi_1 + d_2\phi_2 \mid -q^{-1} \mid \frac{\lambda_1-a}{\lambda_2-a} \mid d_2 \leq d_1 \leq q^{-1} \mid \frac{b-\lambda_1}{b-\lambda_2} \mid d_2 \right\},$$

where $d_2 \geq 0$. And let

$$R_4^* = \left\{ d_1\phi_1 + d_2\phi_2 \mid -q^{-1} \mid \frac{\lambda_1-a}{\lambda_2-a} \mid |d_2| \leq d_1 \leq q^{-1} \mid \frac{b-\lambda_1}{b-\lambda_2} \mid |d_2| \right\},$$

where $d_2 \leq 0$. Then the union of R_1, R_2^*, R_3, R_4^* is the plane V .

To investigate a relation between the multiplicity of solutions and source terms in a nonlinear elliptic differential equation

$$Au + bu^+ - au^- = h \quad \text{in } H,$$

we consider the restriction $\Pi|_{C_i}$ ($1 \leq i \leq 4$) of Π to the cone C_i . Let $\Pi_i = \Pi|_{C_i}$, i.e.,

$$\Pi_i : C_i \rightarrow V.$$

We have investigated next theorem in [3]

THEOREM 2.3. *For $i = 2, 4$, if we let $\Pi_i(C_i) = R_i$, then R_2 is one of sets $R_1 \cup R_4^*$ or $R_2^* \cup R_3$, and R_4 is one of sets $R_3 \cup R_4^*$ or $R_1 \cup R_2^*$. Furthermore the restriction Π_i maps C_i onto R_i .*

3. Critical points and multiplicity results

We investigate the multiplicity of solutions of a nonlinear elliptic differential equation

$$(3.1) \quad Au + bu^+ - au^- = t\phi_1 \quad \text{in } H,$$

where $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t > 0$.

Above all, We will investigate using critical point theory that $R_2 = R_1 \cup R_4^*$ and $R_4 = R_1 \cup R_2^*$.

Henceforth, let F denote the functional defined by

$$(3.2) \quad F(u) = \int_{\Omega} \left[\frac{1}{2} |\nabla u|^2 - G(u) + t\phi_1 u \right] dx,$$

where $G(u) = \frac{1}{2} (b(u^+)^2 + a(u^-)^2)$ and $u \in E$. Then,

$$DF(u)y = F'(u)y = \int_{\Omega} (\nabla u \cdot \nabla y - g(u)y + t\phi_1 y) dx \quad \text{for all } y \in E$$

and solutions of (3.1) coincide with solutions of

$$(3.3) \quad DF(u) = 0,$$

where $g(u) = G'(u) = bu^+ - au^-$.

Therefore, we shall investigate critical points of F . We know the following theorem.

THEOREM 3.1. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3, h \in V$. Let $v \in V$ be given. Then there exists a unique solution $z \in W$ of the equation*

$$(3.4) \quad Az + (I - P)(b(v + z)^+ - a(v + z)^- - h) = 0 \quad \text{in } W.$$

If $z = \theta(v)$, then θ is continuous on V and we have $DF(v + \theta(v))(w) = 0$ for all $w \in W$. In particular $\theta(v)$ satisfies a uniform Lipschitz in v with respect to the L^2 -norm. If $\tilde{F} : V \rightarrow R$ is defined by $\tilde{F}(v) = F(v + \theta(v))$, then $\tilde{F}$ has continuous Frechét derivative $D\tilde{F}$ with respect to v and

$$D\tilde{F}(v)(r) = DF(v + \theta(v))(r) \quad \text{for all } r \in V.$$

If v_0 is a critical point of $\tilde{F}$, then $v_0 + \theta(v_0)$ is a solution of (3.1) and conversely every solution of (3.1) is $D\tilde{F}(v_0) = 0$.

THEOREM 3.2. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Then we have:*

(a) *Let $t = b - \lambda_1$ ($h = (b - \lambda_1)\phi_1$). Then equation (3.1) has a positive solution v_p and there exists a small open neighborhood B_p of v_p in C_1 such that in B_p , v_p is a strict local point of maximum of $\tilde{F}$.*

(b) *$t = \lambda_1 - a$ ($h = (\lambda_1 - a)\phi_1$). Then equation (3.1) has a negative solution v_n and there exists a small open neighborhood B_n of v_n in C_3 such that in B_n , v_n is a saddle point of $\tilde{F}$.*

Proof. (a) Let $t = b - \lambda_1$ ($h = (b - \lambda_1)\phi_1$). Then equation (3.1) has a $u_p = \phi_1$ which is of the form $u_p = v_p + \theta(v_p)$. (in this case $\theta(v_p) = 0$) and $I + \theta$, where I is an identity map on V , is continuous. Since v_p is in the interior of C_1 , there exists a small open neighborhood B_p of v_p in C_1 . We note that $\theta(v) = 0$ in B_p . Therefore, if $v = v_p + v^* \in B_p$, then we have

$$\begin{aligned} \tilde{F}(v) &= \tilde{F}(v_p + v^*) \\ &= \int_{\Omega} \left[\frac{1}{2} (|\nabla(v_p + v^*)|^2 - b((v_p + v^*)^+)^2 - a((v_p + v^*)^-)^2) \right. \\ &\quad \left. + h(v_p + v^*) \right] dx \\ &= \frac{1}{2} \int_{\Omega} (|\nabla v^*|^2 - b v^{*2}) dx + \int_{\Omega} [\nabla v_p \cdot \nabla v^* - b v_p v^* + h v^*] dx \\ &\quad + \int_{\Omega} \left[\frac{1}{2} (|\nabla v_p|^2 - b v_p^2) + h v_p \right] dx \\ &= \frac{1}{2} \int_{\Omega} (|\nabla v^*|^2 - b v^{*2}) dx + \int_{\Omega} [\nabla v_p \cdot \nabla v^* - b v_p v^* + h v^*] dx + C, \end{aligned}$$

where $C = \int_{\Omega} \left[\frac{1}{2}(|\nabla v_p|^2 - bv_p^2) + hv_p \right] dx = F(u_p) = \tilde{F}(v_p)$.

If $v \in V$ and $v = c_1\phi_1 + c_2\phi_2$, then we have

$$(3.7) \quad \begin{aligned} \|v\|_0^2 &= \int_{\Omega} |\nabla v|^2 dx = \sum_{i=1}^2 c_i^2 \lambda_i < \lambda_2 \sum_{i=1}^2 c_i^2 \\ &= \lambda_2 \int_{\Omega} v^2 dx = \lambda_2 \|v\|^2 \end{aligned}$$

Let $v^* = c_1\phi_1 + c_2\phi_2$ and let $v = v_p + v^* \in B_p$. Then

$$\int_{\Omega} [\nabla v_p \cdot \nabla v^* - bv_p v^* + hv^*] dx = 0.$$

By (3.7),

$$\tilde{F}(v) - \tilde{F}(v_p) = \frac{1}{2} \int_{\Omega} (|\nabla v^*|^2 - bv^{*2}) dx < (\lambda_2 - b) \int_{\Omega} v^2 dx.$$

Since $\lambda_2 < b$, it follows that for $t = b - \lambda_1$, v_p is a strict local point of maximum for $\tilde{F}(v)$.

(b) Let $t = \lambda_1 - a$ ($h = (\lambda_1 - a)\phi_1$). Then equation (3.1) has a negative solution $u_n = -\phi_1$ which is of the form $u_n = v_n + \theta(v_n)$, where $\theta(v_n)$ and $-I + \theta$ is continuous in V . Since v_n is the interior, $\text{Int}C_3$, of C_3 . We note that $\theta(v) = 0$ in B_n . Therefore, if $v = v_n + v_* \in B_n$, then we have to calculate

$$\begin{aligned} \tilde{F}(v) - \tilde{F}(v_n) &= \frac{1}{2} \int_{\Omega} (|\nabla v_*|^2 - av_*^2) dx \\ &= \frac{1}{2} (c_1^2(\lambda_1 - a) + c_2^2(\lambda_2 - a)). \end{aligned}$$

The above equation implies that v_n is a saddle point of $\tilde{F}$. □

Therefore, by Theorem 3.2 and [7], we can obtain the following theorem.

THEOREM 3.3. *Let $h \in V$ and let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. For fixed t the functional $\tilde{F}$, defined on V , satisfies the Palais-Smale condition : Any sequence $\{v_n\}_1^\infty \subset V$ for which $\tilde{F}(v_n)$ is bounded and $D\tilde{F}(v_n) \rightarrow 0$ possesses a convergent subsequence.*

Let $\hat{V}$ be the vector space spanned by an eigenfunction ϕ_2 . Let $\hat{W}$ denote the orthogonal complement of $\hat{V}$ and let $\hat{P} : H \rightarrow \hat{V}$ denote the orthogonal projection of H onto $\hat{V}$. By the use of (3.1), (3.2) and Theorem 3.1, we have the following statements.

Given $\hat{v} \in \hat{V}$ and $t \in \mathbf{R}$, there exists a unique solution $\hat{z} = \hat{\theta}(\hat{v})$ of

$$A\hat{z} + (I - \hat{P})g(\hat{v} + \hat{z}) = t\phi_1, \hat{z}|_{\partial\Omega} = 0,$$

where $\hat{z} \in \hat{W}$.

If $\hat{z} = \hat{\theta}(\hat{v})$, then $\hat{\theta}$ is continuous on $\hat{V}$. Let $\hat{F}_0(\hat{v})$ denote the functional defined by $\hat{F}_0(\hat{v}) = F(\hat{v} + \hat{\theta}(\hat{v}))$. Then $\hat{F}_0$ has a continuous Frechét derivative $D\hat{F}_0$ with respect to $\hat{v}$ and u is a solution of equation (3.1) if and only if $u = \hat{v} + \hat{\theta}(\hat{v})$ and $D\hat{F}_0(\hat{v}) = 0$, where $\hat{v} = \hat{P}u$. By Theorem 3.3, for each fixed t the functional $\hat{F}_0$ satisfies the Palais-Smale condition.

By Theorem 3.1, the functional $\hat{F}_0(\hat{v})$ satisfy the following lemma.

LEMMA 3.4. *If $t > 0$ there exists $\alpha = \alpha(t) > 0$ such that if $\hat{v} \in \hat{V}$ and $\|\hat{v}\|_0 < \alpha(t)$, then $\hat{\theta}(\hat{v}) = t\phi_1/(b - \lambda_1)$ for $t > 0$ and the point $\hat{v} = 0$ is a strict local point of maximum for $\hat{F}_0$.*

LEMMA 3.5. *For $k > 0$ and $t = 0$, $\hat{F}_0(k\hat{v}) = k^2\hat{F}_0(\hat{v})$.*

Proof. Since g is positively homogeneous of degree one, it follows that

if $\hat{v} \in \hat{V}$, $\hat{z} \in \hat{W}$ and $A\hat{z} + (I - \hat{P})g(\hat{v} + \hat{z}) = 0$, $\hat{z}|_{\partial\Omega} = 0$, then

$A(k\hat{z}) + (I - \hat{P})g(k\hat{v} + k\hat{z}) = 0$. Therefore, $\hat{\theta}(k\hat{v}) = k\hat{\theta}(\hat{v})$. We see that $F_0(ku) = k^2F(u)$ for $u \in H$ and $k > 0$. Hence, $\hat{F}_0(k\hat{v}) = F(k\hat{v} + \hat{\theta}(k\hat{v})) = k^2F(\hat{v} + \hat{\theta}(\hat{v})) = k^2\hat{F}_0(\hat{v})$. $\square$

LEMMA 3.6. Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Then we have:

(a) For $t = 0$, $\hat{F}_0(\hat{v}) > 0$ for all $\hat{v} \in \hat{V}$ with $\hat{v} \neq 0$.

(b) For $t > 0$, $\hat{F}_0(\hat{v}) \rightarrow \infty$ as $\|\hat{v}\|_0 \rightarrow \infty$.

(c) For fixed $t > 0$, $\tilde{F}(v) \rightarrow \infty$ along a ϕ_2 -axis.

Proof. With Lemma 3.5 and [7], we have (a) and (b).

(c) For fixed t we see that $F(\hat{v} + \hat{\theta}(\hat{v})) = F(v + \theta(v))$. Let $\tilde{F}|_{\hat{V}}$ be the restriction of $\tilde{F}$ to the $\hat{V}$. Then $\tilde{F}|_{\hat{V}} = \hat{F}_0$. By (b), if $t > 0$, then $\tilde{F}(v) \rightarrow \infty$ as along a ϕ_2 -axis. $\square$

LEMMA 3.6. Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1$ and $q^2 \mid \lambda_2 - a \mid > \mid \lambda_1 - a \mid$. Then we have $\tilde{F}(v) \rightarrow +\infty$ as $\|v\|_0 \rightarrow \infty$ along a boundary ray of C_3 .

Proof. Let $v = v_p + v_* \in C_3$ and $v_* = c_1\phi_1 + c_2\phi_2$. Then we have

$$\begin{aligned} \tilde{F}(v) &= \int_{\Omega} \left[\frac{1}{2} (|\nabla(v_p + v_*)|^2 - a((v_p + v_*)^-)^2) \right. \\ &\quad \left. + (b - \lambda_1)\phi_1(v_p + v_*) \right] dx. \end{aligned}$$

We note that $v_p + v_* \in \partial C_3$ if and only if $c_2 = q(c_1 + 1)$, $c_1 \leq -1$. It can be shown easily the following holds

$$\begin{aligned} \tilde{F}(v) &= \frac{1}{2} ((\lambda_1 - a)c_1^2 + q^2(\lambda_2 - a)c_1^2) \\ &\quad + (q^2(\lambda_2 - a) + (b - a))c_1 + \frac{1}{2} ((\lambda_2 - a)q^2 + (b - a)) + C, \end{aligned}$$

where $C = \int_{\Omega} \left[\frac{1}{2} (|\nabla v_p|^2 - b v_p^2) + (b - \lambda_1)\phi_1 v_p \right] dx$. Hence if $v \in \partial C_3$, then we have $\tilde{F}(v) \rightarrow +\infty$ as $c_1 \rightarrow -\infty$. $\square$

THEOREM 3.8. Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1$. Then $\tilde{F}(v)$ has a critical point in $\text{Int}C_1$, and at least one critical point in $\text{Int}C_2$, and at least one critical point in $\text{Int}C_4$.

Proof. We denote that $-\tilde{F}(v) = \tilde{F}_*(v)$. By Theorem 3.2 (a), if $t = b - \lambda_1$, then there exists a small open neighborhood B_p of v_p in C_1 such that in B_p , $v_p = \phi_1$ is a strict local point of maximum for $\tilde{F}(v)$. Hence v_p is a strict local point of minimum for $\tilde{F}_*(v)$ in C_1 . By Lemma 3.6 (c), $\tilde{F}_*(v) \rightarrow -\infty$ as $\|v\|_0 \rightarrow \infty$ along a ϕ_2 -axis. and $\tilde{F}_* \in C^1(V, \mathbf{R})$ satisfies the Palais-Smale condition.

Since $\tilde{F}_*(v) \rightarrow -\infty$ as $\|v\|_0 \rightarrow \infty$ along a ϕ_2 -axis, we can choose v_0 on ϕ_2 -axis such that $\tilde{F}_*(v_0) < \tilde{F}_*(v_p)$. Let Γ be the set of all paths in V joining v_p and v_0 . We write

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_*(v).$$

The fact that in B_p , v_p is a strict local point of minimum of $\tilde{F}_*$, the fact that $\tilde{F}_*(v) \rightarrow -\infty$ as $\|v\|_0 \rightarrow \infty$ along a ϕ_2 -axis, the fact $\tilde{F}_*$ satisfies the Palais-Smale condition, and the Mountain Pass Theorem imply that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_*(v)$$

is a critical value of $\tilde{F}_*$ (see Mountain Pass Theorem and [1, 7]). When $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1$, equation (3.1) has a unique positive solution v_p and no negative solution. Hence there exists a critical point v_3 , in $\text{Int}(C_2 \cup C_4)$, of $\tilde{F}_*$ such that

$$\tilde{F}_*(v_3) = c.$$

We prove that if $v_3 \in \text{Int}C_4$ such that $\tilde{F}_*(v_3) = c$, then there exists another critical point $v \in \text{Int}C_2$ of $\tilde{F}_*$.

Suppose $v_3 \in \text{Int}C_4$. Since $\tilde{F}_*(v) \rightarrow -\infty$ as $\|v\|_0 \rightarrow \infty$ along a ϕ_2 -axis, we can choose v_1 on this ϕ_2 -axis such that $\tilde{F}_*(v_1) < \tilde{F}_*(v_p)$. Let Γ_1 be the set of all paths in $C_1 \cup C_2 \cup C_3$ joining v_p and v_1 . We write

$$c' = \inf_{\gamma \in \Gamma_1} \sup_{\gamma} \tilde{F}_*(v).$$

We note that $\tilde{F}_*(v) \rightarrow \infty$ as $\|v\|_0 \rightarrow \infty$ along a negative ϕ_1 -axis or along a boundary ray, $c_2 = q(c_1 + 1)(c_1 \geq -1)$, of C_1 , where $v = v_p + c_1\phi_1 + c_2\phi_2 \in \partial C_1$.

Let us fix ε, η as in Deformation Lemma with $E = V, F = \tilde{F}_*, c = c', K_{c'} = \phi$ and taking $\varepsilon < \frac{1}{2}(c' - \tilde{F}_*(v_p))$. Taking $\gamma \in \Gamma_1$ such that $\sup_\gamma \tilde{F}_* \leq c'$. From Deformation lemma([3]), $\eta(1, \cdot) \circ \gamma \in \Gamma_1$ and

$$\sup \tilde{F}_*(\eta(1, \cdot) \circ \gamma) \leq c' - \varepsilon < c',$$

which is a contradiction. Therefore there exists a critical point v_4 of $\tilde{F}_*$ at level c' such that $v_4 \in C_1 \cup C_2 \cup C_3$ and $\tilde{F}_*(v_4) = c'$. Since equation (3.1) has a unique positive solution v_p and no negative solution when $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1 (> 0)$, the critical point v_4 belongs to $\text{Int}C_2$.

Similarly, we have that if $v_3 \in \text{Int}C_2$ with $\tilde{F}_*(v_3) = c$, then $\tilde{F}_*(v)$ has another critical point in $\text{Int}C_4$. The critical point of $\tilde{F}_*$ if and only if the critical point of $\tilde{F}$. Hence this completes the theorem. $\square$

THEOREM 3.9. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. For $1 \leq i \leq 4$, let $\Pi(C_i) = R_i$. Then $R_2 = R_1 \cup R_4^*$ and $R_4 = R_1 \cup R_2^*$.*

Proof. Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $h \in V$. We note that v is a solution of the equation

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = h \text{ in } V$$

if and only if v is a critical point of $\tilde{F}$. Hence it follows from Theorem 3.8 that $R_2 \cap R_1 \neq \emptyset$. Since R_2 is one of sets $R_1 \cup R_4^*$ or $R_3 \cup R_2^*$, R_2 must be $R_1 \cup R_4^*$.

On the other hand, it follows from Theorem 3.8 that $R_4 \cap R_1 \neq \emptyset$. Since R_4 is one of sets $R_1 \cup R_2^*$ or $R_3 \cup R_4^*$, R_4 must be $R_1 \cup R_2^*$. $\square$

By Theorem 2.2, Theorem 2.3 and Theorem 3.9, we obtain the main theorem of this section.

THEOREM 3.10. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Then we have the following.*

(a) *If $h \in \text{Int}R_1$, then equation (1.2) has a positive solution and at least two change sign solutions.*

(b) *If $h \in \partial R_1$, then equation (1.2) has a positive solution and at least one change sign solution.*

(c) *If $h \in \text{Int}R_i^*$ ($i = 2, 4$), then equation (1.2) has at least one change sign solution.*

(d) *If $h \in \text{Int}R_3^*$, then equation (1.2) has only the negative solution.*

(e) *If $h \in \partial R_3$, then equation (1.2) has a negative solution.*

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