WEAK CONVERGENCE OF VARIOUS MODELS TO FRACTIONAL BROWNIAN MOTION

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Abstract. We consider arrival process and ON/OFF source model which allows for long packet trains and long inter-train distances. We prove the weak convergence of these processes to Fractional Brownian motion. Finally, we figure out the coefficients of $B_H(t)$ and time $t$ when ON/OFF periods have the Pareto distribution.

1. Introduction

Many researchers have studied long range dependent process and self-similar process because of burstiness of network traffic at any time scale. Though the various models proposed for capturing the long-range dependent nature of network traffic are all either exactly or asymptotically second order self-similar, their effect on network performance can be very different ([6], [7], [8]).

Self-similarity, long range dependence and heavy tailed process have been observed in many time series, i.e. network traffic and finance ([4]). In particular, fractional Brownian motion and FARIMA in modern packet network traffic has been the focus of much attention ([5]). Various methods for estimating the self-similar parameter and intensity of long range dependence in time series has been investigated ([7], [9]). And, there has been a recent flood of literature and discussion on the tail behavior of queue-length distribution, motivated by potential applications to the design and control by high-speed telecommunication networks ([1], [2], [3]).

In this paper we consider arrival process based on autoregressive process and show that the suitably scaled distributions of these processes

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converge to fractional Brownian motion in the sense of finite dimensional distributions.

On the other hand, we consider idealized ON/OFF source model which allows for long packet trains and long inter-train distances. [9] proved that the aggregate cumulative packet process behaves like linear combination of fractional Brownian motion $B_H(t)$ and time $t$. When ON/OFF periods have the Pareto distribution, we figure out the coefficients of $B_H(t)$ and time $t$.

In section 2, we define the short range dependence, long range dependence, fractional Brownian motion and self-similarity. In section 3, we prove the weak convergence of arrival process and autoregressive process to fractional Brownian motion. In section 4, we figure out the coefficients of $B_H(t)$ and time $t$ when ON/OFF periods have the Pareto distribution.

2. Definition and Preliminary

In this section we first define short range dependence and long range dependence. Let $\tau_X(k)$ be the covariance of stationary stochastic process $X(t)$.

**Definition 2.1.** A stationary stochastic process $X(t)$ exhibits short range dependence if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| < \infty$$

**Definition 2.2.** A stationary stochastic process $X(t)$ exhibits long range dependence if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| = \infty$$

A standard example of a long range dependent process is fractional Brownian motion, with Hurst parameter $H > \frac{1}{2}$.

**Definition 2.3.** A stochastic process $\{B_H(t)\}$ is said to be a Fractional Brownian motion (FBM) with Hurst parameter $H$ if

1. $B_H(t)$ has stationary increments
2. for $t > 0$, $B_H(t)$ is normally distributed with mean 0
3. $B_H(0) = 0$ a.s.
4. The increments of $B_H(t)$, $Z(j) = B_H(j + 1) - B_H(j)$ satisfy

$$\rho_Z(k) = \frac{1}{2}\{|k + 1|^{2H} + |k - 1|^{2H} - 2k^{2H}\}$$

Fractional Brownian motion is an important example of self-similar process defined below.

**Definition 2.4.** A continuous process $X(t)$ is self-similar with self-similarity parameter $H \geq 0$ if it satisfies the condition:

$$X(t) \overset{d}{=} c^{-H}X(ct), \quad \forall t \geq 0, \forall c > 0,$$

where the equality is in the sense of finite-dimensional distributions.

3. Weak Convergence to Fractional Brownian motion

Let $X^j(i)$ be the number of arrivals in the $i$th time unit of $j$th source. Let

$$X_M(i) = \sum_{j=1}^{M}(X^j(i) - E(X^j(i)),$$

and $\tau(k)$ denote the covariance of $X_1(i)$.

**Lemma 3.1.** ([4]) The stationary sequence

$$\frac{1}{M^{1/2}}X_M(i)$$

converges in the sense of finite dimensional distributions to $G_H(i)$, where $G_H(i)$ represents a stationary Gaussian process with covariance function of the same form as $\tau(k)$, as $M \to \infty$.

**Theorem 3.1.**

$$\lim_{T \to \infty} \lim_{M \to \infty} \frac{1}{THM^{1/2}} \sum_{i=0}^{[T]} X_M(i)$$

converges to $\{\sigma_0B_H(t)|0 \leq t \leq 1\}$ in the sense of finite dimensional distributions. Furthermore, as $M \to \infty$ and $T \to \infty$,

(a) (Long Range dependence) If $\tau(k) \sim ck^{2H-2}$, $c > 0$ and $1/2 < H < 1$,
then \( \sigma_0^2 = \frac{c}{H(2H - 1)} \).

(b) If

\[
\sum_{k=1}^{\infty} |\tau(k)| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \tau(k) = c > 0,
\]

then \( \sigma_0^2 = c \).

(c) (Short Range dependence)

\[
\tau(k) \sim ck^{2H-2}, \quad c < 0 \quad \text{and} \quad 0 < H < 1/2,
\]

then \( \sigma_0^2 = -\frac{c}{H(2H - 1)} \).

**Proof.** Set \( Z_i = 1/M^{1/2}X_M(i) \). By Lemma 3.1, \( Z_i \) converges in the sense of finite dimensional distributions to \( G_H(i) \) as \( M \) goes to infinity. By Theorem 7.2.11 of [5], the finite dimensional distributions of \( T^{-H} \sum_{i=0}^{[Tt]} Z_i \) converges to those of \( \{\sigma_0B_H(t), 0 \leq t \leq 1\} \).

**THEOREM 3.2.** Let \( X_t \) be the autoregressive process of order one, i.e. \( X_t = \phi_1X_{t-1} + a_t \), where \( a_t \sim N(0,1) \) for each \( t \). Then

\[
\lim_{T \to \infty} \lim_{M \to \infty} \sum_{i=0}^{[Tt]} X_M(i) = \sqrt{\frac{\phi_1}{1 - \phi_1}} B(t),
\]

where, \( B(t) \) is a Brownian Motion.

**Proof.** We know that

\[
(1 - \phi_1B)X_t = a_t,
\]

i.e.

\[
X_t = \sum_{i=0}^{\infty} \phi_1^i a_{t-i}.
\]

And, we get

\[
\text{Cov}_{X_t}(k) = \phi_1^k, \quad k \geq 1, \quad |\phi_1| < 1.
\]

Therefore,

\[
\tau(k) = \phi_1^k,
\]

for large \( M \). Since

\[
\sum \tau(k) = \sum \phi_1^k = \frac{\phi_1}{1 - \phi_1} < \infty,
\]
we get, by Theorem 3.1 (b),

$$\lim_{T \to \infty} \lim_{M \to \infty} \sum_{i=0}^{[Tt]} X_M(i) = \frac{\phi_1}{1 - \phi_1} B_{1/2}(t) = \frac{\phi_1}{1 - \phi_1} B(t).$$

\[

4. Convergence of ON/OFF Source Model

Let us consider the stationary time series \( \{X(t), t \geq 0\} \). \( X(t) = 1 \) means that there is a packet at time \( t \) and \( X(t) = 0 \) means that there is no packet. Viewing \( X(t) \) as the reward at time \( t \), we have a reward of 1 throughout on ON-period, then a reward of 0 throughout the following OFF-periods, then 1 again, and so on. Suppose the lengths of the ON-periods are i.i.d., those of the OFF-periods are i.i.d. and the lengths of ON-periods and OFF-periods are independent. But the ON-periods and OFF-periods may have the different distributions.

Suppose that there are \( M \) i.i.d. sources. Since each source sends its own sequence of packet trains, it has its own reward sequence \( \{X^{(m)}(t)\} \). Therefore, the cumulative packet count at time \( t \) is

$$\sum_{m=1}^{M} X^{(m)}(t).$$

Rescaling time by a factor \( T \), we consider the aggregated cumulative packet counts

$$X_M(Tt) = \int_0^{Tt} \left( \sum_{m=1}^{M} X^{(m)}(u) \right) du$$

in the interval \([0, Tt]\). To specify the distributions of ON-period \( O_1 \) and OFF-periods \( O_2 \), let

$$\mu_1 = EO_1, \mu_2 = EO_2$$

and as \( x \to \infty \), tailing distributions of \( O_1, O_2 \) are

$$l_1 x^{-\alpha_1} L_1(x) \quad \text{and} \quad l_2 x^{-\alpha_2} L_2(x)$$

with \( 1 < \alpha_j < 2 \), where is a constant \( l_j > 0 \) and \( L_j > 0 \) is a slowly varying function at infinity.
Notation. When $1 < \alpha_j < 2$, set
\[ a_j = l_j(\Gamma(2 - \alpha_j))/(\alpha_j - 1), \]
\[ b = \lim_{t \to \infty} t^{\alpha_2 - \alpha_1} \frac{L_1(t)}{L_2(t)}. \]
If $0 < b < \infty$ then set
\[ \sigma^2 = \frac{2(\mu_2^3 a_1 b + \mu_1^3 a_2)}{(\mu_1 + \mu_2)^3 \Gamma(4 - \alpha_{\text{min}})}. \]
if $b = 0$ or $b = \infty$ then set
\[ \sigma^2 = \frac{2\mu_{\text{max}}^2 a_{\text{min}}}{(\mu_1 + \mu_2)^3 \Gamma(4 - \alpha_{\text{min}})}. \]

**Lemma 4.1.** For large $M$ and $T$, the aggregate packet process
\[ \{X_M(Tt), t \geq 0\} \]
behaves statistically like
\[ TM \frac{\mu_1}{\mu_1 + \mu_2} t + T^H \sqrt{L(t)M} \sigma_B H(t) \]
where $H = (3 - \alpha_{\text{min}})/2$ and $\sigma$ is as above.

*Proof.* Theorem 1 of [9]

Suppose that ON/OFF periods $O_j$ has the Pareto distribution
\[ P(O_j > x) = K^{\alpha_j}x^{-\alpha_j} \text{ for } x \geq K > 0. \]
When $1 < \alpha_j < 2$, each periods has infinite variance.

**Theorem 4.1.** Let $O_j$ be ON/OFF-periods that has the Pareto distributions as above. Then, for large $M$ and $T$, the aggregate packet process \[ \{X_M(Tt), t \geq 0\} \]
behaves statistically like
\[ TM \frac{\alpha_1 \alpha_2 - \alpha_1}{2\alpha_1 \alpha_2 - \alpha_1 - \alpha_2} t + T^H \sigma_B H(t) \]
where, $H = (3 - \alpha_{\text{min}})/2$ .

**Case 1.** Suppose that $O_j$ have the same distributions, i.e., $\alpha_1 = \alpha_2 = \alpha$, then
\[ H = \frac{3 - \alpha}{2}. \]
and
\[ \sigma^2 = \frac{K^{\alpha-1}\Gamma(2-\alpha)}{2\alpha\Gamma(4-\alpha)} \]

Case 2. If \( \alpha_1 < \alpha_2 \), then
\[ H = \frac{3 - \alpha_1}{2} \]
and
\[ \sigma^2 = \frac{2K^2\alpha_1^2(\alpha_1 - 1)(\alpha_2 - 1)^3a_{\min}}{(2\alpha_1\alpha_2K - \alpha_1K - \alpha_2K)^3} \]

Case 3. If \( \alpha_1 > \alpha_2 \), then
\[ H = \frac{3 - \alpha_2}{2} \]
and
\[ \sigma^2 = \frac{2K^2\alpha_2^2(\alpha_2 - 1)(\alpha_1 - 1)^3a_{\min}}{(2\alpha_1\alpha_2K - \alpha_1K - \alpha_2K)^3} \]

Proof. Since the expectation of the Pareto distribution is
\[ \frac{\alpha_jK}{\alpha_j - 1} \]
for \( j = 1, 2, \cdots \). By Lemma 4.1, the coefficient of time \( t \) is
\[ \frac{\alpha_1\alpha_2 - \alpha_1}{2\alpha_1\alpha_2 - \alpha_1 - \alpha_2} \].

Case 1. Since \( O_j \) have the same distributions, we get
\[ \lim_{t \to \infty} t^{\alpha_2 - \alpha_1} = 1. \]
And we know
\[ \alpha_1 = \alpha_2 = K^\alpha\Gamma(2-\alpha)/(\alpha - 1). \]
Thus, we get
\[ \sigma^2 = \frac{K^{\alpha-1}\Gamma(2-\alpha)}{2\alpha\Gamma(4-\alpha)}. \]

In the similar way, we can get Case 2 and Case 3. \(\Box\)
References


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