Korean J. Math. **33** (2025), No. 2, pp. 13–21
https://dx.doi.org/10.11568/kjm.2025.33.2.13

BOUND FOR THE ZEROS OF QUATERNIONIC POLYNOMIAL WITHOUT RESTRICTIONS

Bilal Dar*,† and Abdul Liman

Abstract. In this paper, we are concerned with the problem of locating the zeros of regular polynomials of quaternionic variable without any restriction on the coefficients. We derive new bounds for the zeros of these polynomials by virtue of a maximum modulus theorem and the structure of the zero sets in the newly developed theory of regular functions and polynomials of a quaternionic variable. With no restriction on the coefficients, our results provide new bound for the zeros of quaternionic polynomials in a four dimensional space.

1. Introduction

The study of polynomial zeros indeed has a rich history in mathematics and has played a crucial role in the development of various mathematical concepts and theories. The quest for understanding the roots of polynomials has led to the formulation of algebraic techniques and tools, making it a fundamental part of contemporary algebra. Theoretical research inspired by polynomial zeros has contributed significantly to the broader field of mathematics. Notably, the development of algebraic methods and the study of algebraic structures emerged from the investigation of polynomial equations and their solutions. This exploration laid the foundation for abstract algebra, which encompasses the study of algebraic structures such as groups, rings, and fields. The concept of limiting polynomials introduces a useful perspective when dealing with the zeros of a polynomial. This approach can be beneficial when algebraic and analytic methods face challenges in determining the roots. By considering the behaviour of the polynomial as certain parameters approach limiting values, researchers can gain valuable information about the distribution and characteristics of the zeros. Putting restrictions on the coefficients of a polynomial can be a fruitful approach for achieving better bounds. These restrictions may take various forms, such as constraints on the magnitudes, signs, or relationships among the coefficients. By introducing such limitations, researchers aim to simplify the problem and obtain more manageable and

Received February 19, 2024. Revised December 25, 2024. Accepted February 5, 2025.
2010 Mathematics Subject Classification: 30E10, 30G35, 16K20.
Key words and phrases: Quaternions, Zeros, Polynomials.

† The author is working in J&K Higher Education Department and is presently posted at Govt. Degree College Pampore. The author is thankful to J&K Govt. for granting permission to pursue Ph.D. under part-time basis at NIT Srinagar.
∗ Corresponding author.

insightful results regarding the location and behaviour of the polynomial zeros. In this direction, the following elegant result on the location of zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya Theorem (see [4], [12], [13]). G. Eneström appears to have been the first to obtain a finding of this sort while researching a problem in pension fund theory. S. Kakeya [11] presented a paper in the Tôhoku Mathematical Journal in 1912 that featured the following more comprehensive result:

Theorem 1.1. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $0 < a_0 \leq a_1 \leq ... \leq a_n$*, then all the zeros of* p *lie in* $|z| \leq 1$.

In the literature, for example see ([1], [8], [10], [12], [13]), there exist various extensions and generalizations of Eneström-Kakeya Theorem. In 1967, Joyal, Labelle, and Rahman [10] published a result which might be considered the foundation of the studies which we are currently studying. The Eneström-Kakeya Theorem, as stated in Theorem 1.1, deals with polynomials with non-negative coefficients which form a monotone sequence. Joyal, Labelle, and Rahman generalized Theorem 1.1 by dropping the condition of non-negativity and maintaining the condition of monotonicity. Namely, they proved:

Theorem 1.2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $a_0 \leq a_1 \leq ... \leq a_n$*, then all the zeros of* p *lie in* $|z| \leq \frac{1}{|a_n|}(|a_0| + a_n - a_0)$.

2. Preliminary

In the recent study (for example, see [3], [5] [6], [2], [14]), the development of a new theory of regularity for functions, particularly focusing on polynomials of a quaternionic variable, represents a significant advancement in mathematical research. This theory seems to extend the study of functions beyond the realm of complex variables, specifically to quaternionic variables. The extension to quaternionic variables is interesting because quaternions, unlike complex numbers, have more than one imaginary unit and exhibit richer mathematical structures. The mentioned theory appears to draw parallels between the properties of holomorphic functions in the complex plane and the regular functions in the quaternionic space. Holomorphic functions in the complex plane are well studied, and their zero sets play a crucial role in understanding their behavior. The preliminary steps of the theory involve describing the structure of the zero sets of quaternionic regular functions and investigating the factorization property of zeros. Understanding the distribution and behaviour of zeros is a crucial aspect of studying functions, and factorization properties provide insights into how functions can be decomposed based on their zeros.
It is worth noting that quaternionic analysis introduces additional challenges compared to complex analysis due to the non-commutativity of quaternion multiplication. As a result, extending concepts from complex analysis to quaternionic analysis often requires careful consideration and adaptation of mathematical techniques. In this regard, Gentili and Stoppato [5] gave a necessary and sufficient condition for a quaternionic regular function to have zero at a point in terms of the coefficients of the power series expansion of the function. Before we state our results, we need to introduce some preliminaries on quaternions and quaternionic polynomials.

Quaternions: Quaternions are the extension of complex numbers to four dimensions, introduced by William Rowan Hamilton in 1843. The set of all quaternions is denoted by $\mathbb{H}$ in honour of Sir Hamilton and are generally represented in the form $q = \alpha + i\beta + j\gamma + k\delta \in \mathbb{H}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and i, j, k are the fundamental quaternion units, such that $i^2 = j^2 = k^2 = ijk = -1$. Each quaternion q has a conjugate. The conjugate of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is denoted by $\overline{q}$ and is defined as $\overline{q} = \alpha - i\beta - j\gamma - k\delta$. Moreover, the norm (or length) of a quaternion q is given by

$$||q|| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

The quaternions are the standard example of a non-commutative division ring and also form a four-dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis.
The indeterminate for a quaternionic polynomial is defined as q. Without commutativity, we have the polynomial aq^n and the polynomial $a_0qa_1q\cdots qa_n$, $a = a_0a_1\cdots a_n$. To address this issue, we use the standard that polynomials have indeterminate on the left and coefficients on the right, resulting in the quaternionic polynomial $p_1(q) = \sum\limits_{s=0}^{m} q^s a_s$. For such p_1 and $p_2(q) = \sum\limits_{s=0}^{n} q^s b_s$, the regular product of p_1 and p_2 is defined as $\big(p_1 * p_2\big)(q) = \sum\limits_{i,j=0}^{n,m} q^{i+j} a_i b_j$. This is consistent with the definition of the regular product for the power series of a quaternionic variable (see definition 3.1 of [5]). If p_1 has real coefficients, then $*$ multiplication is equivalent to point-wise multiplication. In general, the product rule $*$ is associative rather than commutative. We define the set of quaternionic polynomials by

$$\mathscr{P}_n := \left\{ p \ ; \ p(q) = \sum_{l=0}^{n} q^l a_l \ , \ q \in \mathbb{H} \right\}$$

where $a_l \in \mathbb{H}$ or $\mathbb{R}$, $0 \le l \le n$.
Indeed, the absence of commutativity in quaternions introduces distinct behaviours for polynomials compared to their counterparts in the real or complex cases. The Factor Theorem, which states that a being a zero of $p(z)$ is equivalent to $(z - a)$ being a divisor of $p(z)$, relies on the commutativity of the underlying ring. In the case of real or complex polynomials, the ring of coefficients is commutative, allowing for the straightforward application of the Factor Theorem (see Theorem III. 6.6 of [7]). The non-commutativity of quaternion multiplication results in a departure from the familiar behaviour of polynomials. In the quaternionic case, the factor Theorem, as traditionally stated, may not hold in the same way.
In quaternionic analysis, zero sets of quaternionic polynomials can exhibit different characteristics compared to their real or complex counterparts. The lack of commutativity introduces complexities in the factorization properties of polynomials, and the relationship between zeros and divisors may not be as straightforward. This non-commutative behaviour adds an extra layer of intricacy to the study of polynomials in quaternionic analysis, making it a fascinating and challenging area of research. Researchers working in quaternionic analysis often need to develop new tools and techniques to understand the behaviour of quaternionic polynomials and their zero sets. In the Quaternion case, the second degree polynomial $q^2 + 1$ has an infinite number of zeros, namely $q_0 = i \ orj \ ork$ and all those given by $w_0 = h^{-1}q_0h \ \forall h \in \mathbb{H}$.

It will be interesting to locate all zeros of a quaternionic polynomial. In this direction, Carney et al. [3] proved the following extension of Theorem 1.1 for the quaternionic polynomial $p \in \mathscr{P}_n$. More precisely, they proved the following result:

THEOREM 2.1. *If $p \in \mathscr{P}_n$ is a quaternionic polynomial of degree n with real coefficients satisfying $0 < a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of p lie in $|q| \leq 1$.*

In the same paper, they proved the following result which replaces the condition of monotonicity on the real coefficients by monotonicity in the real and imaginary parts of the quaternion coefficients:

THEOREM 2.2. *If $p \in \mathscr{P}_n$ is a quaternionic polynomial of degree n where $a_l = \alpha_l + \beta_l i + \gamma_l j + \delta_l k \in \mathbb{H}$; $0 \leq l \leq n$ and*

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \ ; \ \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$$
$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n \ ; \ \delta_0 \leq \delta_1 \leq \cdots \leq \delta_n$$

then all the zeros of p lie in

$$|q| \leq \frac{\big(|\alpha_0| - \alpha_0 + a_n\big) + \big(|\beta_0| - \beta_0 + \beta_n\big) + \big(|\gamma_0| - \gamma_0 + \gamma_n\big) + \big(|\delta_0| - \delta_0 + \delta_n\big)}{|a_n|}.$$

The development of a new bound for the zeros of quaternionic polynomials, especially without imposing restrictions on the coefficients, indicates progress in understanding the behaviour of these polynomials. To obtain a better bound than what is provided by existing theorems, such as Theorem 2.2, suggests advancements in the precision and accuracy of characterizing the distribution of zeros in quaternionic polynomials. This could have implications for various applications in mathematics and related fields where quaternionic polynomials arise. The absence of restrictions on the coefficients is noteworthy, as it implies that the bound holds for a broader class of quaternionic polynomials. This generality enhances the applicability of the results and may contribute to a deeper understanding of the relationships between the coefficients and the zeros in quaternionic polynomials. As such, we provide a new bound for the zeros of quaternionic polynomials without any restriction on the coefficients.

3. Main Results

THEOREM 3.1. *All zeros of the quaternion polynomial $p \in \mathscr{P}_n$ of degree n where $a_l = \alpha_l + \beta_l i + \gamma_l j + \delta_l k \in \mathbb{H}$; $0 \leq l \leq n$ lie in*

$$|q| \leq \frac{1}{|a_n|}\left(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + n^{\frac{p-1}{p}} A_p\right)$$

where $A_p = \left(\sum\limits_{s=1}^{n} \Big|a_s - a_{s-1}\Big|^p\right)^{\frac{1}{p}}$ and $p \geq 1$.

Letting $p \to \infty$ in Theorem 3.1 and noting that

$$\lim_{p\to\infty} \left(\sum_{s=1}^{n} \big|a_s - a_{s-1}\big|^p\right)^{\frac{1}{p}} = \max_{1 \leq s \leq n} \big|a_s - a_{s-1}\big|,$$

we obtain the following result:

Theorem 3.2. *All zeros of the quaternion polynomial $p \in \mathscr{P}_n$ of degree n where $a_l = \alpha_l + \beta_l i + \gamma_l j + \delta_l k \in \mathbb{H}$; $0 \le l \le n$ lie in*

$$|q| \le \frac{1}{|a_n|}\Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + nM\Big),$$

where $M = \max_{1 \le s \le n} \big|a_s - a_{s-1}\big|$.
Taking $p = 1$ in Theorem 3.1, we get the following result:

Theorem 3.3. *All zeros of the quaternion polynomial $p \in \mathscr{P}_n$ of degree n where $a_l = \alpha_l + \beta_l i + \gamma_l j + \delta_l k \in \mathbb{H}$; $0 \le l \le n$ lie in*

$$|q| \le \frac{1}{|a_n|}\Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + A_1\Big)$$

where $A_1 = \sum_{s=1}^{n} \Big|a_s - a_{s-1}\Big|$.
From the definition of M and A_1, we obtain $A_1 \le nM$, therefore, Theorem 3.3 provides a better bound than Theorem 3.2.
If we take in Theorem 3.3 all coefficients real i.e, $a_l = \alpha_l$, $0 \le l \le n$ satisfying $a_n \ge a_{n-1} \ge \cdots \ge a_0$ so that $\beta_l = \gamma_l = \delta_l = 0 \; \forall \; l = 0, 1, 2, \cdots, n$, we get the following quaternion analogue of Theorem 1.2:

Theorem 3.4. *All zeros of the quaternion polynomial $p \in \mathscr{P}_n$ of degree n with real coefficients satisfying $a_n \ge a_{n-1} \ge \cdots \ge a_0$ lie in*

$$|q| \le \frac{1}{|a_n|}\Big(|\alpha_0| + a_n - a_0\Big).$$

Obviously for $a_0 > 0$, Theorem 3.4 reduces to Theorem 2.1.
Next if we assume for $0 \le l \le n$

$$\begin{aligned}
&\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_l \; ; \; \beta_n \ge \beta_{n-1} \ge \cdots \ge \beta_l \\
&\gamma_n \ge \gamma_{n-1} \ge \cdots \ge \gamma_l \; ; \; \delta_n \ge \delta_{n-1} \ge \cdots \ge \delta_l,
\end{aligned} \tag{1}$$

so that

$$\begin{aligned}
\Big|a_s - a_{s-1}\Big| &= \Big|(\alpha_s - \alpha_{s-1}) + i(\beta_s - \beta_{s-1}) + j(\gamma_s - \gamma_{s-1}) + k(\delta_s - \delta_{s-1})\Big| \\
&\le |\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|
\end{aligned}$$

then, we obtain

$$\begin{aligned}
A_1 &\le \sum_{s=1}^{n} \Big[|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|\Big] \\
&= \sum_{s=l+1}^{n} \Big[|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|\Big] \\
&+ \sum_{s=1}^{l} \Big[|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|\Big] \\
&= \big(\alpha_n - \alpha_l\big) + \big(\beta_n - \beta_l\big) + \big(\gamma_n - \gamma_l\big) + \big(\delta_n - \delta_l\big) \\
&+ \sum_{s=1}^{l} \Big[|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|\Big].
\end{aligned} \tag{2}$$

Inequality (2) demonstrates that Theorem 3.3 provides a refinement of the recent result by D. Tripathi [15] [Theorem 3.1].

4. Lemmas

In order to prove the Theorem 3.1, we need the following lemmas:
Lemma 1. Let f and g be given quaternionic power series with radii of convergence greater than R and let $q_0 \in B(0,\ R)$. Then $\big(f * g\big)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\big(f(q_0)^{-1}q_0 f(q_0)\big) = 0$.
Above lemma is due to Gentili et al. [5].
Lemma 2. Let $B = B(0,\ R)$ be a ball in $\mathbb{H}$ with centre 0 and radius R and let $f : B \to \mathbb{H}$ be a regular function. if $|f|$ has a relative maximum at a point $a \in B$ then f is constant on B.
Above lemma is due to Gentili et al. [6].
Following Lemma is known result deduced from Jensen's inequality [9] for generalized mean of positive real numbers.
Lemma 3. If λ_l ; $i = 1, 2, \cdots, n$ are non-negative numbers then for $0 < t \leq p$

$$\left(\frac{1}{n}\sum_{s=1}^{n}\lambda_i^t\right)^{\frac{1}{t}} \leq \left(\frac{1}{n}\sum_{s=1}^{n}\lambda_i^p\right)^{\frac{1}{p}}.$$

5. Proof of Theorem

Proof of Theorem 3.1

Proof. We have,

$$\begin{aligned} p(q) * (1-q) &= \big(q^n a_n + q^{n-1}a_{n-1} + \cdots + qa_1 + a_0\big) * (1-q) \\ &= -q^{n+1}a_n + q^n(a_n - a_{n-1}) + q^{n-1}(a_{n-1} - a_{n-2}) + \cdots + q(a_1 - a_0) + a_0 \\ &= f(q) - q^{n+1}a_n \end{aligned} \tag{3}$$

where $f(q) = a_0 + \sum\limits_{s=1}^{n} q^s(a_s - a_{s-1})$. By Lemma 1, $p(q) * (1-q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $1 - p(q)^{-1}qp(q) = 0$. Notice that $1 - p(q)^{-1}qp(q) = 0$ is equivalent to $p(q)^{-1}qp(q) = 1$ and if $p(q) \neq 0$, this implies that $q = 1$. So the only zeros of $p(q) * (1-q) = 0$ are zeros of $p(q)$ and $q = 1$. Applying Lemma 3 with $t = 1$, we have for $|q| = 1$ and $p \geq 1$

$$\begin{aligned}
|f(q)| &\leq \left|a_0 + \sum_{s=1}^{n} q^s(a_s - a_{s-1})\right| \\
&\leq |a_0| + \sum_{s=1}^{n} \left|q^s\right|\left|a_s - a_{s-1}\right| \\
&= |a_0| + n\left(\frac{1}{n}\sum_{s=1}^{n}\left|a_s - a_{s-1}\right|\right) \\
&\leq |a_0| + n\left(\frac{1}{n}\sum_{s=1}^{n}\left|a_s - a_{s-1}\right|^p\right)^{\frac{1}{p}} \\
&= \left|\alpha_0 + \beta_0 i + \gamma_0 j + \delta_0 k + \right| + n^{\frac{p-1}{p}}\left(\sum_{s=1}^{n}\left|a_s - a_{s-1}\right|^p\right)^{\frac{1}{p}} \\
&\leq \left|\alpha_0\right| + \left|\beta_0\right| + \left|\gamma_0\right| + \left|\delta_0\right| + n^{\frac{p-1}{p}} A_p, \qquad (4)
\end{aligned}$$

where $A_p = \left(\sum_{s=1}^{n}\left|a_s - a_{s-1}\right|^p\right)^{\frac{1}{p}}$.

We have

$$\max_{|q|=1}\left|q^n * f(\frac{1}{q})\right| = \max_{|q|=1}\left|q^n f(\frac{1}{q})\right| = \max_{|q|=1}\left|f(\frac{1}{q})\right| = \max_{|q|=1}\left|f(q)\right|.$$

Therefore $|q^n * f(\frac{1}{q})|$ has same bound on $|q| = 1$ as of $|f(q)|$, hence we obtain from inequality (4)

$$\left|q^n * f(\frac{1}{q})\right| \leq \left|\alpha_0\right| + \left|\beta_0\right| + \left|\gamma_0\right| + \left|\delta_0\right| + n^{\frac{p-1}{p}} A_p \ for \ |q| = 1.$$

By Maximum Modulus Theorem, we have

$$\left|q^n * f(\frac{1}{q})\right| = \left|q^n f(\frac{1}{q})\right| \leq \left|\alpha_0\right| + \left|\beta_0\right| + \left|\gamma_0\right| + \left|\delta_0\right| + n^{\frac{p-1}{p}} A_p \ for \ |q| \leq 1.$$

This implies,

$$|q^n|\left|f(\frac{1}{q})\right| \leq \left|\alpha_0\right| + \left|\beta_0\right| + \left|\gamma_0\right| + \left|\delta_0\right| + n^{\frac{p-1}{p}} A_p \ for \ |q| \leq 1.$$

Replacing q by $\frac{1}{q}$, we obtain

$$|f(q)| \leq \left(\left|\alpha_0\right| + \left|\beta_0\right| + \left|\gamma_0\right| + \left|\delta_0\right| + n^{\frac{p-1}{p}} A_p\right)|q|^n \ for \ |q| \geq 1. \qquad (5)$$

With the help of inequality (5), we obtain from equation (3) that for $|q| \geq 1$

$$\begin{aligned}\big|p(q) * (1-q)\big| &= \big|f(q) - q^{n+1}a_n\big| \\ &\geq |q|^{n+1}|a_n| - \big|f(q)\big| \\ &\geq |q|^{n+1}|a_n| - \Big(\big|\alpha_0\big| + \big|\beta_0\big| + \big|\gamma_0\big| + \big|\delta_0\big| + n^{\frac{p-1}{p}}A_p\Big)|q|^n \\ &= |q|^n\Bigg[|q||a_n| - \Big(\big|\alpha_0\big| + \big|\beta_0\big| + \big|\gamma_0\big| + \big|\delta_0\big| + n^{\frac{p-1}{p}}A_p\Big)\Bigg] \\ &> 0\end{aligned}$$

if

$$|q| > \frac{1}{|a_n|}\Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + n^{\frac{p-1}{p}}A_p\Big).$$

This implies, $p(q) * (1-q) \neq 0$ if

$$|q| > \frac{1}{|a_n|}\Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + n^{\frac{p-1}{p}}A_p\Big).$$

This gives, $p(q) * (1-q) = 0$ if

$$|q| \leq \frac{1}{|a_n|}\Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + n^{\frac{p-1}{p}}A_p\Big).$$

Since the only zeros of $p(q) * (1-q)$ are $q = 1$ and the zeros of $p(q)$, it follows that all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + n^{\frac{p-1}{p}}A_p\Big).$$

This completes the proof of Theorem 3.1. □

References

[1] A. Aziz, B. A. Zargar, *Some extension of Eneström-Kakeya theorem*, Glasńik Mathematiki., **31**, 239-2441, (1996).
https://web.math.pmf.unizg.hr/glasnik/vol_31/vol31no2.html

[2] L. Brand, *The Roots of a Quaternion*, Amer. Math. Monthly. 49(8), (1942), 519-520.
https://doi.org/10.1080/00029890.1942.11991274

[3] N. Carney, R. Gardner, R. Keaton and A. powers, *The Eneström-Kakeya theorem of a quaternionic variable*, J. Approx. Theory, **250** (2020), 105-325.
https://doi.org/10.1016/j.jat.2019.105325

[4] G. Eneström, *Remarque sur un théoréme relatif aux racines de I'équation $a_nx^n + ... + a_0 = 0$ oú tous les coefficients sont réels et positifs*, Tôhoku Math. J., **18** (1920), 34-36.

[5] G. Gentili and C. Stoppato, *Zeros of Regular Functions and Polynomials of a Quaternionic variable*, Michigan Math. J., 56 (2008), 655-667.
https://doi.org/10.1307/mmj/1231770366

[6] G. Gentili and D. Struppa, *A new theory of regular functions of a Quaternionic variable*, Adv. Math., 216 (2007), 279-301.
https://doi.org/10.1016/j.aim.2007.05.010

[7] G. Gentili, D. Struppa, *On the multiplicity of zeros of polynomials with quaternionic coefficients*, Milan J. Math., **76** (2008), 15-25.
https://doi.org/10.1007/s00032-008-0093-0
[8] N. K. Govil, Q. I. Rahman, *On the Eneström-Kakeya theorem,* Tôhoku Math. J., **20** (1968), 126-136.
[9] J.L.W.V. Jensen, *Sur les fonctions convexes et les ingalits les valeurs moyennes*, Acta Mathematica, **30** (1) (1906), 175-193.
https://doi.org/10.1007/BF02418571
[10] A. Joyal, G. Labelle, Q. I. Rahman, *On the location of zeros of polynomials*, Can. Math. Bull., **10** (1967), 53-63.
https://doi.org/10.4153/CMB-1967-006-3
[11] S. Kakeya, *On the limits of the roots of an algebraic equation with positive coefficients*, Tôhoku Math. J., **2** (1912–13), 140–142.
[12] M. Marden, *Geometry of polynomials*, Math. Surveys, **3** (1966).
[13] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in polynomials: Extremal problems, Inequalities, Zeros,* World Scientific publishing Co., Singapore, (1994).
https://doi.org/10.1142/1284
[14] S. Mir, A. Liman, *Some extensions of Enström Kakeya Theorem for Quaternionic Polynomials*, Korean J. Math., **30** (2022), No. 4, 615-628.
http://dx.doi.org/10.11568/kjm.2022.30.4.615
[15] D. Tripathi, *A note on Eneström-Kakeya theorem for a polynomial with quaternionic variable*, Arab. J. Math., **9** (2020), 707-714.
https://doi.org/10.1007/s40065-020-00283-0

Bilal Dar
Department of Mathematics, National Institute of Technology,
Srinagar 190006, India
E-mail: darbilal67@gmail.com

Abdul Liman
Department of Mathematics, National Institute of Technology,
Srinagar 190006, India
E-mail: abliman@rediffmail.com