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A FOURTH-ORDER ITERATIVE BOUNDARY VALUE PROBLEM WITH CONJUGATE BOUNDARY CONDITIONS

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Abstract. We establish conditions on the function f for the existence and uniqueness of solutions for the fourth-order iterative differential equation

$$x^{(4)}(t) = f(t, x(t), x^{[2]}(t), ..., x^{[m]}(t)), \quad a < t < b$$

$m \geq 2$, with solutions subject to one of the boundary conditions

$$x(a) = c,\ x'(a) = 0,\ x''(a) = 0,\ x(b) = d,$$
$$x(a) = c,\ x'(a) = 0,\ x(b) = d,\ x'(b) = 0,$$
$$x(a) = c,\ x(b) = d,\ x'(b) = 0,\ x''(b) = 0.$$

We assume that a, b, c, d are constants such that $a < c < d < b$. The main tool employed is Schauder's Fixed-Point Theorem.

1. Introduction

The study of state-dependent differential equations has a long history with application in climate models, economic models, electrodynamical systems, infectious disease models, mechanics, neural networks, population dynamics, and various other fields. See, for example, [2, 4, 6, 8, 9, 13, 14] and the references therein. Iterative differential equations are a special case of state-dependent differential equations. For a sampling of results on iterative differential equations, see [3, 5, 7, 10–12, 15, 17] and references therein.

The goal of this current work is to broaden the results found from Kaufmann and Whaley [12]. In that manuscript, the authors gave sufficient conditions for the existence and uniqueness of solutions for the fourth-order iterative boundary value problem,

$$x^{(4)}(t) = f(t, x(t), x^{[2]}(t), ..., x^{[m]}(t)), \qquad a < t < b \tag{1.1}$$

$m \geq 2$, that satisfy one of the following sets of boundary conditions

$$x(-a) = -a,\ x'(-a) = b,\ x''(-a) = c,\ x(a) = a, \tag{1.2}$$

$$x(-a) = -a,\ x(a) = a,\ x'(a) = b,\ x''(a) = c. \tag{1.3}$$

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Here $x^{[2]}(t) = x\big(x(t)\big)$, and for $j = 3, ..., m, x^{[j]}(t) = x\big(x^{[j-1]}\big)(t)$. To ensure that $x : [-a, a] \to [-a, a]$, the authors restricted f to be of one sign and put bounds on f so that the equivalent integral operators of (1.1), (1.2) and (1.1), (1.3) were monotonic.

In this paper we consider existence and uniqueness of solutions for the fourth-order iterative problem (1.1), with solutions satisfying one of the boundary conditions:

$$x(a) = c,\ x'(a) = 0,\ x''(a) = 0,\ x(b) = d, \tag{1.4}$$

$$x(a) = c,\ x'(a) = 0,\ x(b) = d,\ x'(b) = 0, \tag{1.5}$$

$$x(a) = c,\ x(b) = d,\ x'(b) = 0,\ x''(b) = 0, \tag{1.6}$$

where $a < c < d < b$. We assume throughout that $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous. We allow f to change signs and, by using a modification of the technique in [11], show directly that the associated integral operators map $[a, b]$ into $[a, b]$. As a consequence, the iterates $x^{[j]}(t), j = 2, 3, \dots,$ are well-defined.

The manuscript is organized as follows. In Section 2, we will rewrite (1.1), (1.4) as an integral equation, and provide conditions under which the solution of the integral equation will be a solution of the boundary value problem. We will also list properties of the Green's function and of the norm of the difference of two iterative functions. In Section 3, we will state and prove results concerning the existence and uniqueness of solutions of (1.1), (1.4). In Section 4, we present the equivalent inversion of (1.1), (1.5) and (1.1), (1.6) and state, without proof, the analogous existence and uniqueness results. Examples are included to illustrate our results.

2. Preliminaries

Our first goal of this section is to rewrite (1.1), (1.4) as an integral equation. We will accomplish this by first inverting the non-homogeneous equation with homogeneous boundary conditions, and then solving the homogeneous equation with non-homogeneous boundary conditions. The inversion of (1.1), (1.4) will be the sum of the two expressions. Later in this section, we state two key lemmas that will be used sections 3 and 4. One will involve the Green's function and the other deals with the norm of the difference of iterates. Lastly, we state a version of Schauder's fixed point theorem.

We will begin the inversion by considering

$$x^{(4)}(t) = g(t), \quad a \le t \le b, \tag{2.1}$$

$$x(a) = x'(a) = x''(a) = x(b) = 0. \tag{2.2}$$

Using the same method found in [12], it can be shown that x is a solution to (2.1), (2.2), if, and only if, x satisfies the integral equation

$$x(t) = \int_a^b G(t, s) g(s)\, ds \tag{2.3}$$

where

$$G(t, s) = \frac{-1}{6(b-a)^3} \begin{cases} (b-s)^3(t-a)^3 - (b-a)^3(t-s)^3, & a \le s \le t \le b, \\ (b-s)^3(t-a)^3, & a \le t \le s \le b. \end{cases} \tag{2.4}$$

It is easy to show that if x is a solution of,

$$x^{(4)}(t) = 0,$$
$$x(a) = c,\ x'(a) = 0,\ x''(a) = 0,\ x(b) = d,$$

then x is given by

$$x(t) = \frac{d-c}{(b-a)^3}(t-a)^3 + c. \tag{2.5}$$

Consequently, if x is a solution of (1.1), (1.4), then x will then be the sum of (2.3) and (2.5). That is, x is a solution of the integral equation

$$\begin{aligned} x(t) &= \int_a^b G(t,s) f(s, x(s), x^{[2]}(s), ..., x^{[m]}(s))\, ds \\ &\quad + c + \frac{d-c}{(b-a)^3}(t-a)^3, \end{aligned} \tag{2.6}$$

where $G(t,s)$ is given in (2.4).

In order for solutions of the boundary value problems to be well-defined, we also require the image of x be in the interval $[a,b]$; that is, in order for $x(x^{[m]})(t)$ to be defined, we need $a \le x(t) \le b$ for all $t \in [a,b]$. Knowing this, we can show that if $x \in C[a,b]$, satisfies $a \le x(t) \le b$ for all t, and satisfies the integral equation (4.2), then it satisfies (1.1), (1.4). This gives us the following lemma.

Lemma 2.1. *The function $x \in C^4[a,b]$ is a solution of* (1.1), (1.4) *if and only if $x \in C[a,b]$ satisfies $a \le x(t) \le b$ and the integral equation*

$$\begin{aligned} x(t) &= \int_a^b G(t,s) f(s, x(s), x^{[2]}(s), ..., x^{[m]}(s))\, ds \\ &\quad + \frac{d-c}{(b-a)^3}(t-a)^3 + c, \end{aligned}$$

where $G(t,s)$ is defined in (2.4).

To prove the existence and uniqueness of solutions of (1.1), (1.4), we will need to bound our Green's functions $G(t,s)$. For that, we will use the following lemma.

Lemma 2.2. *The Green's function given in* (2.4) *satisfies the following inequality:*

$$0 \le |G(t,s)| \le \frac{(b-a)^3}{3}.$$

Proof. First note that $g_1(t) = (b-s)^3(t-a)^3, a \le t \le s \le b$ is an increasing function of t. So $(b-s)^3(t-a)^3 \le (b-s)^3(s-a)^3$. We know that $\max_{s\in[a,b]}(b-s)^3(s-a)^3$ occurs at $s = \frac{a+b}{2}$, and is given by $\frac{(b-a)^6}{64}$. Thus, $\frac{(b-s)^3(t-a)^3}{6(b-a)^3} \le \frac{(b-a)^3}{6\cdot 64} \le \frac{(b-a)^3}{3}$.

Now, consider $g_2(s) = (b-s)^3(t-a)^3 - (b-a)^3(t-s)^3, a \le s \le t \le b$. Then, $|g_2(s)| \le (b-s)^3(t-a)^3 + (b-a)^3(t-s)^3 \le (b-a)^6 + (b-a)^6 = 2(b-a)^6$. Hence, $\frac{1}{6(b-a)^3}\left|(b-s)^3(t-a)^3 - (b-a)^3(t-s)^3\right| \le \frac{1}{6(b-a)^3}\cdot 2(b-a)^6 = \frac{(b-a)^3}{3}$. Consequently, $|G(t,s)| \le \frac{(b-a)^3}{3}$ □

We use the Banach space $\Phi = (C[a,b], \|\cdot\|)$, where the norm is given by $\|x\| = \max_{t\in[a,b]}|x(t)|$. Define the operator $T_1 : C[a,b] \to C[a,b]$ by

$$(T_1x)(t) = \int_a^b G(t,s)f(s,x(s),x^{[2]}(s),...,x^{[m]}(s))\,ds + \frac{d-c}{(b-a)^3}(t-a)^3 + c \tag{2.7}$$

where $G(t,s)$ is defined in (2.4). We will also need the subspace

$$\Phi(J,N) = \{x \in \Phi : \|x\| \le J, |x(t_2) - x(t_1)| \le N|t_1 - t_2|, t_1, t_2 \in [a,b]\},$$

as well as the following lemma, the proof of which can be found in [16], [18].

Lemma 2.3. *If $x, y \in \Phi(J,N)$, then*

$$\left|x^{[m]}(t_1) - x^{[m]}(t_2)\right| \le N^m|t_1 - t_2|, m = 0,1,2,...,$$

for all $t_1, t_2 \in [a,b]$ and

$$\left\|x^{[m]}(t_1) - x^{[m]}(t_2)\right\| \le \sum_{j=0}^{m-1} N^j \|x - y\|.$$

We end this section by stating a version of Schauder's fixed point theorem which can be found in [1].

Theorem 2.4 (Schauder). *Let A be a nonempty compact convex subset of a Banach space and let $T : A \to B$ be continuous, where B is a compact subset of A. Then T has a fixed point in B.*

3. Existence and Uniqueness Results for (1.1), (1.4)

In this section we will state and prove our existence and uniqueness results for (1.1), (1.4). Let $T_1 : C[a,b] \to C[a,b]$ be defined as (2.7). Throughout the section we will assume the following conditions hold.

(H1) There exists $\alpha_\ell \in L[a,b], \ell = 1,2,\ldots,m$, such that

$$|f(t,x_1,\ldots,x_m) - f(t,y_1,\ldots,y_m)| \le \sum_{\ell=1}^{m} \alpha_\ell(t)\|x_\ell - y_\ell\|$$

for all $t \in [a,b]$ and $x_i, y_i \in \mathbb{R}, i = 1,2,\ldots,m$.

(H2) There exists a $K_1 \in \mathbb{R}$ such that $0 < K_1 < \min\left\{\frac{12(b-d)}{(b-a)^4}, \frac{12(c-a)}{(b-a)^4}\right\}$ and

$$-K_1 \le f\big(t,u_1,u_2,...,u_m\big) \le K_1$$

for all $t \in [a,b]$, and for all $u_i \in \mathbb{R}, i = 1,2,...,m$.

Theorem 3.1. *Suppose that condition (H1) and (H2) holds. Then there exists a solution to* (1.1), (1.4).

Proof. Let $R = \max\{|a|,|b|\}$. Consider the convex, compact nonempty set $\Phi(R,N)$, where

$$N = \frac{3|d-c|}{(b-a)} + K_1\left(\frac{17(b-a)^3}{12}\right). \tag{3.1}$$

We first need to prove that $T_1 : \Phi(R,N) \to \Phi(R,N)$. We do this by first showing that $a \le (T_1x)(t) \le b$ for all $t \in [a,b]$. By $(H2)$, we have

$$\begin{aligned}
(T_1x)(t) &= -\frac{(t-a)^3}{6(b-a)^3}\int_a^b (b-s)^3 f(s)\,ds + \frac{1}{6}\int_a^t (t-s)^3 f(s)\,ds \\
&\quad + \frac{d-c}{(b-a)^3}(t-a)^3 + c \\
&\le \frac{K_1}{6(b-a)^3}(t-a)^3\int_a^b (b-s)^3\,ds + \frac{K_1}{6}\int_a^t (t-s)^3\,ds \\
&\quad + \frac{d-c}{(b-a)^3}(t-a)^3 + c \\
&\le \frac{K_1(b-a)}{24}(t-a)^3 + \frac{K_1}{24}(t-a)^4 + \frac{d-c}{(b-a)^3}(t-a)^3 + c \\
&\le K_1\frac{(b-a)^4}{12} + d.
\end{aligned}$$

Using the inequality in $(H2)$, we see that.

$$K_1\frac{(b-a)^4}{12} + d \le \frac{12(b-d)}{(b-a)^4}\frac{(b-a)^4}{12} + d \le b$$

for all $t \in [a,b]$. That is, $(Tx)(t) \le b$. A similar argument can be used to show that $(Tx)(t) \ge a$. Therefore, for all $t \in [a,b]$, $a \le (Tx)(t) \le b$. Consequently,

$$|T(x(t))| \le \max\{|a|,|b|\} = R \tag{3.2}$$

To complete the proof that $T_1 : \Phi(R,N) \to \Phi(R,N)$, we need to show that for given $t_1, t_2 \in [a,b]$, $\big|(T_1x)(t_2) - (T_1x)(t_1)\big| \le N|t_2 - t_1|$, where N is defined as above. We may assume, without loss of generality, that $t_2 \le t_1$. To this end, first note that

$$\begin{aligned}
&|(T_1x)(t_2) - (T_1x)(t_1)| \\
&\qquad = \int_a^b \big|G(t_2,s) - G(t_1,s)\big|\,\big|f\big(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s)\big)\big|\,ds \\
&\qquad\quad + \frac{(d-c)}{(b-a)^3}\big((t_2-a)^3 - (t_1-a)^3\big) \\
&\qquad\quad K_1\int_a^b \big|G(t_2,s) - G(t_1,s)\big|\,ds + \frac{3(d-c)}{(b-a)}|t_2 - t_1|.
\end{aligned}$$

We need to bound $\int_a^b |G(t_2,s) - G(t_1,s)|\,ds$ by a constant time $|t_2 - t_1|$. Since $t_2 \le t_1$, we can rewrite the integral as

$$\begin{aligned}
\int_a^b \big|G(t_2,s) - G(t_1,s)\big|\,ds &\le \int_a^{t_1} \big|G(t_2,s) - G(t_1,s)\big|\,ds \\
&\quad + \int_{t_1}^{t_2} \big|G(t_2,s) - G(t_1,s)\big|\,ds \\
&\quad + \int_{t_2}^{b} \big|G(t_2,s) - G(t_1,s)\big|\,ds.
\end{aligned}$$

We consider each term separately.

Given that $t_1 \le t_2$, we have using the definition of $G(t,s)$ that the first term on the right satisfies

$$\begin{aligned}
\int_a^{t_1} |G(t_2,s) &- G(t_1,s)|\,ds \\
&\le \frac{1}{6(b-a)^3}\int_a^{t_1} \big|(b-s)^3\big((t_2-a)^3-(t_1-a)^3\big)\big| \\
&\quad + (b-a)^3\big|(t_2-s)^3-(t_1-s)^3\big|\,ds \\
&\le \frac{1}{2(b-a)}\left(\frac{(b-a)^4}{4}-\frac{(b-t_1)^4}{4}\right)|t_2-t_1| \\
&\quad + \frac{(b-a)^3}{2}|t_2-t_1| \\
&\le \left[\frac{(b-a)^3}{8}+\frac{(b-a)^3}{2}\right]|t_2-t_1| \\
&= \frac{5(b-a)^3}{8}|t_2-t_1|.
\end{aligned}$$

From Lemma 2.2, we obtain,

$$\begin{aligned}
\int_{t_1}^{t_2} \big|G(t_2,s)-G(t_1,s)\big|\,ds &\le \int_{t_1}^{t_2} |G(t_2,s)-G(t_1,s)|\,ds \\
&\le \int_{t_1}^{t_2} 2\frac{(b-a)^3}{3}\,ds \\
&= \frac{2(b-a)^3}{3}|t_2-t_1|.
\end{aligned}$$

And finally,

$$\begin{aligned}
\int_{t_2}^{b} |G(t_2,s) &- G(t_1,s)|\,ds \\
&\le \frac{1}{6(b-a)^3}\int_{t_2}^{b} \big|(b-s)^3\big((t_2-a)^3-(t_1-a)^3\big)\big|\,ds \\
&\le \frac{1}{6(b-a)^3}\left(\frac{(b-t_2)^4}{4}(3(b-a)^2)\right)|t_2-t_1| \\
&\le \frac{(b-a)^3}{8}|t_2-t_1|.
\end{aligned}$$

Thus,

$$\int_a^b |G(t_2,s)-G(t_1,s)|\,ds \le \frac{17(b-a)^3}{12}|t_2-t_1|.$$

Consequently, we have,

$$\begin{aligned}
|(T_1x)(t_2)-(T_1x)(t_1)| &\le \left(\frac{|3(d-c)|}{(b-a)}+K_1\frac{17(b-a)^3}{12}\right)|t_2-t_1| \\
&= N|t_2-t_1|. \qquad (3.3)
\end{aligned}$$

and so, $T_1 : \Phi(R,N) \to \Phi(R,N)$.

From (3.2) and (3.3) and an application of the Arzelà-Ascoli Theorem, the set $T_1(\Phi(R,N))$ is precompact. It follows from $(H1)$ that,

$$|(T_1x)(t) - (T_1y)(t)| \le \frac{2(b-a)^3}{3} \int_a^b \sum_{\ell=1}^{m} \alpha_\ell(s)\, ds \|x^{[\ell]} - y^{[\ell]}\|.$$

From here, standard methods can be used to show that T_1 is continuous. Hence, by Theorem 2.4 there exists a fixed point of the operator T_1. According to Lemma (2.1), this fixed point is a solution of (1.1), (1.4) □

Example 3.2. *Consider the following boundary value problem with parameter k.*

$$x^{(4)}(t) = kt^2 \cos(x^{[2]}(t)) \tag{3.4}$$

$$x(0) = \frac{\pi}{3},\ x'(0) = x''(0) = 0,\ x(\pi) = \frac{\pi}{2}. \tag{3.5}$$

Here, $a = 0, b = \pi, c = \frac{\pi}{3}$, and $d = \frac{\pi}{2}$. Notice $a < c < b$ and $a < d < b$. Let $\alpha_1(t) = 0$ and $\alpha_2(t) = kt^2$. Then,

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le \alpha_2(t)|x_2 - y_2|$$

for all $t \in [0, \pi]$. Also, $0 \le f(t, x, x^{[2]}) \le k\pi^2 = K_1$. By Theorem 3.1, if $K_1 < \min\left\{\frac{12(b-d)}{(b-a)^4}, \frac{12(c-a)}{(b-a)^4}\right\} = \frac{12(c-a)}{(b-a)^4} = \frac{4}{\pi^3}$ then there exists a solution of (3.2), (3.3)*. Since $K_1 = k\pi^2$, we have that there exists a solution of* (3.2), (3.3) *for all $0 < k < \frac{4}{\pi^5} \approx 1.30711 \times 10^{-2}$.*

We are now ready for our uniqueness result.

Theorem 3.3. *Suppose that $(H1)$ and $(H2)$ hold and that*

$$\frac{2(b-a)^3}{3} \sum_{\ell=1}^{m} \int_a^b \alpha_\ell(s)\, ds \sum_{k=0}^{\ell-1} N^k < 1. \tag{3.6}$$

Then, there exists a unique solution to (1.1), (1.4).

Proof. By Theorem 3.1 and Lemma 2.1, there exists a solution of (1.1), (1.4), which is a fixed point of the mapping T defined in (2.7). Assume x and y are two distinct solutions of (1.1), (1.4). Then, by $(H2)$ and Lemma 2.3, we have for all $t \in [a, b]$,

$$\begin{aligned}
\|x - y\| &= |(Tx)(t) - (Ty)(t)| \\
&= \left| \int_a^b G(t,s) f(s, x, \ldots)\, ds - \int_a^b G(t,s) f(s, y, \ldots)\, ds \right| \\
&\le \int_a^b |G(t,s)||f(s, x, \ldots) - f(s, y, \ldots)|\, ds \\
&\le \frac{2(b-a)^3}{3} \int_a^b \sum_{\ell=1}^{m} \alpha_\ell(s) \|x^{[\ell]} - y^{[\ell]}\|\, ds \\
&\le \left(\frac{2(b-a)^3}{3} \int_a^b \sum_{\ell=1}^{m} \alpha_\ell(s) \sum_{k=0}^{l-1} N^k\, ds \right) \|x - y\|
\end{aligned}$$

where N is given in (3.1). Since (3.6) holds, we arrive at the contradiction

$$\|x - y\| < \|x - y\|.$$

Hence $x = y$, and our fixed point is unique. □

Example 3.4. *To illustrate our uniqueness result we again consider the boundary value problem* (3.2), (3.3). *As in Example* 3.2, $\alpha_1(t) = 0$ *and* $\alpha_2(t) = kt^2$. *The left side of* (3.6) *becomes*

$$\begin{aligned}\frac{2(b-a)^3}{3}\sum_{\ell=1}^{2}\int_a^b \alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}N^k &= \frac{2\pi^3}{3}\int_0^\pi ks^2\,ds\;(1+N)\\ &= \frac{2\pi^6}{9}(1+N)k.\end{aligned}$$

As defined in Theorem 3.1, $N = \frac{3|d-c|}{(b-a)} + K_1\left(\frac{17(b-a)^3}{12}\right)$. *Since* $K_1 = k\pi^2$ $N = \frac{1}{2} + \frac{17\pi^5}{12}k$. *Thus,*

$$\begin{aligned}\frac{2\pi^6}{9}(1+N)k &= \frac{2\pi^6}{9}\left(\frac{3}{2} + \frac{17\pi^5}{12}k\right)k\\ &= \frac{\pi^6}{3}k + \frac{17\pi^{11}}{54}k^2.\end{aligned}$$

According to Theorem 3.3, (3.2), (3.3) *will have a unique solution when* $\frac{\pi^6}{3}k + \frac{17\pi^{11}}{54}k^2 < 1$. *Solving for* k *yields,*

$$\frac{17\pi^{11}}{54}k^2 + \frac{\pi^6}{3}k - 1 < 0.$$

We can apply the quadratic equation to obtain

$$k < \frac{3(\sqrt{102+9\pi} - 3\sqrt{\pi})}{17\pi^{\frac{11}{2}}} \approx 1.98346 \times 10^{-3}.$$

Therefore, there exists a unique solution to (3.2), (3.3).

4. Other Results

In this section we give the corresponding results from Section 3 for the boundary value problems (1.1), (1.5) and (1.1), (1.6). The proof of the results in this section are similar to those found in Section 3. As such, we only point out the main differences in the proof. We begin by considering the boundary value problem (1.1), (1.5).

As in section 2, we can show that $x \in C[a, b]$ is a solution of (1.1), (1.5), if and only if x satisfies the integral equation

$$\begin{aligned}x(t) &= \int_a^b G(t,s)f(s, x(s), x^{[2]}(s), ..., x^{[m]}(s))\,ds\\ &\quad + c + \frac{6(d-c)}{2(b-a)^2}(t-a)^2 - \frac{2(d-c)}{(b-a)^3}(t-a)^3,\end{aligned} \tag{4.1}$$

where

$$G(t,s) = \frac{-1}{6(b-a)^3}\begin{cases}(b-t)^2(a-s)^2(2ab - 3as + ta + bs - 3bt + 2ts), \\ \qquad a \le s \le t \le b, \\ (b-s)^2(a-t)^2(2ab + as - 3ta - 3bs + bt + 2ts), \\ \qquad a \le t \le s \le b.\end{cases} \tag{4.2}$$

Define the operator $T_2 : C[a,b] \to [a,b]$ by

$$(T_2x)(t) = \int_a^b G(t,s)f(s,x(s),x^{[2]}(s),...,x^{[m]}(s))\,ds + c + \frac{6(d-c)}{2(b-a)^2}(t-a)^2 - \frac{2(d-c)}{(b-a)^3}(t-a)^3,$$

where $G(t,s)$ is given as in (4.2). The following lemma dealing with the bound on the Green's function given in (4.2) is straightforward to prove.

LEMMA 4.1. *The Green's function given in* (4.2) *satisfies the following inequality:*

$$0 \le |G(t,s)| \le \frac{(b-a)^3}{3}.$$

In addition to $(H1)$, we will require that the following condition hold.

(H3) There exists a $K_2 \in \mathbb{R}$ such that $0 < K_2 < \min\left\{\frac{12(b-d)}{7(b-a)^4}, \frac{12(c-a)}{7(b-a)^4}\right\}$ and

$$-K_2 < f(t,u_1,u_2,...,u_m) < K_2$$

for all $t \in [a,b]$ and $u_i \in \mathbb{R}$, $i = 1,2,\ldots,m$.

THEOREM 4.2. *Suppose that conditions* $(H1)$ *and* $(H3)$ *hold. Then there exists a solution to* (1.1), (1.6).

Proof. The space needed in this case is $\Phi(R,M)$, where $M = \frac{6(d-c)}{(b-a)} + \frac{6(d-c)(b+a)}{(b-a)^2} + K_2\left(\frac{7(b-a)^3}{24}\right)$. It can then be shown that T_2 maps $\Phi(R,M)$ back into itself and is continuous in a similar manner to Theorem 3.1. This proves the existence of a fixed point, and hence a solution of (1.1), (1.5). □

EXAMPLE 4.3. *Consider again the boundary value problem found in Example 3.2, with the boundary conditions changed to match* (1.1), (1.5).

$$x^{(4)}(t) = kt^2\cos(x^{[2]}(t)) \tag{4.3}$$

$$x(0) = \frac{\pi}{3},\ x'(0) = x'(\pi) = 0,\ x(\pi) = \frac{\pi}{2}, \tag{4.4}$$

Here, $K_2 = k\pi^2$*. So, for all* $0 < K_2 < \frac{12(c-a)}{7(b-a)^4}$*, or* $k < \frac{4}{7\pi^5} \approx 1.86729 \times 10^{-3}$*, there exists a solution to* (4.3), (4.4)*, according to Theorem* 4.2.

We now turn to the uniqueness result.

THEOREM 4.4. *Suppose that* $(H1)$ *and* $(H3)$ *hold and that*

$$\frac{2(b-a)^3}{3}\sum_{\ell=1}^{m}\int_a^b \alpha_\ell(s)\,ds \sum_{k=0}^{\ell-1} N^k < 1. \tag{4.5}$$

Then, there exists a unique solution to (1.1), (1.5).

EXAMPLE 4.5. *To illustrate our uniqueness result, again consider the boundary value problem* (4.3), (4.4)*. Recall from Example* 4.1 *that* $K_2 = k\pi^2$*. The left side of*

(4.5) *becomes*

$$\begin{aligned}\frac{2(b-a)^3}{3}\sum_{\ell=1}^{2}\int_a^b \alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}N^k &= \frac{2\pi^3}{3}\int_0^\pi ks^2\,ds(1+N)\\ &= \frac{2\pi^6}{9}(1+N)k.\end{aligned}$$

As defined in Theorem 4.2, $N = \frac{6(d-c)}{(b-a)} + \frac{6(d-c)(b+a)}{(b-a)^2} + K_2\left(\frac{7(b-a)^3}{24}\right)$. *In this case,* $N = 2 + \frac{7\pi^5}{24}k$. *Thus,*

$$\begin{aligned}\frac{2\pi^6}{9}(1+N)k &= \frac{2\pi^6}{9}\left(3+\frac{7\pi^5}{24}k\right)k\\ &\quad \frac{2\pi^6}{3}k + \frac{7\pi^{11}}{108}k^2.\end{aligned}$$

Preceding in a similar manner to Example 3.4,

$$\begin{aligned}&\frac{7\pi^{11}}{108}k^2 + \frac{2\pi^6}{3}k - 1 < 0\\ &k < \frac{6(\sqrt{51+36\pi}-6\sqrt{\pi})}{7\pi^{\frac{11}{2}}} \approx 1.49385\times 10^{-3}\end{aligned}$$

Therefore, there exists a unique solution to (4.3), (4.4), *according to Theorem 4.4.*

Finally, we give our results for (1.1), (1.6).

Computations similar to Section 2 show that $x(t)$ must satisfy the integral equation

$$\begin{aligned}x(t) &= \int_a^b G(t,s)f(s,x(s),x^{[2]}(s),...,x^{[m]}(s))\,ds\\ &\quad +d+\frac{c-d}{(b-a)^3}(b-t)^3,\end{aligned} \tag{4.6}$$

where

$$G(t,s) = \frac{-1}{6(b-a)^3}\begin{cases}(b-t)^3(s-a)^3, & a\le s\le t\le b,\\ (b-t)^3(s-a)^3-(b-a)^3(s-t)^3, & a\le t\le s\le b.\end{cases} \tag{4.7}$$

The Green's function above enjoys the same bound found in Lemma 2.2.

As done previously, we define the operator $T_3 : C[a,b] \to C[a,b]$ as,

$$\begin{aligned}(T_3x)(t) &= \int_a^b G(t,s)f(s,x(s),x^{[2]}(s),...,x^{[m]}(s))\,ds\\ &\quad +d+\frac{c-d}{(b-a)^3}(b-t)^3,\end{aligned}$$

where $G(t,s)$ is defined as in (4.7).

Beginning with the following assumption in addition to $(H1)$, our existence and uniqueness results for (1.1), (1.6) are as follows.

(H4) There exists a $K_3 \in \mathbb{R}$ such that $0 < K_3 < \min\left\{\frac{12(b-c)}{(b-a)^4}, \frac{12(d-a)}{(b-a)^4}\right\}$ and

$$-K_3 < f(t,u_1,u_2,...,u_m) < K_3$$

for all $t \in [a,b]$ and $u_i \in \mathbb{R}$, $i = 1,2,\ldots,m$.

Theorem 4.6. *Suppose that conditions* $(H1)$ *and* $(H4)$ *hold. Then there exists a solution to* (1.1), (1.6).

Proof. For this proof, the space needed is $\Phi(R, M)$ where $M = \frac{3|c-d|}{(b-a)} + L\left(\frac{17(b-a)^3}{12}\right)$. The rest of the proof follows the same steps as Theorem 3.1. □

Example 4.7. *Once again we look at the boundary value problem,*

$$x^{(4)}(t) = kt^2 \cos(x^{[2]}(t)) \tag{4.8}$$

$$x(0) = \frac{\pi}{3},\ x(\pi) = \frac{\pi}{2},\ x'(\pi) = x''(\pi) = 0. \tag{4.9}$$

With $K_3 = k\pi^2$, *whenever* $0 < K_3 < \frac{12(d-a)}{(b-a)^4} = \frac{6}{\pi^5} \approx 1.96066 \times 10^{-2}$, *there exists a solution to* (4.8), (4.9) *according to Theorem* (4.6).

Theorem 4.8. *Suppose that* $(H1)$ *and* $(H4)$ *hold and that*

$$\frac{2(b-a)^3}{3} \sum_{\ell=1}^{m} \int_a^b \alpha_\ell(s)\, ds \sum_{k=0}^{\ell-1} N^k < 1. \tag{4.10}$$

Then, there exists a unique solution to (1.1), (1.6).

Example 4.9. *Again consider the boundary value problem* (4.8), (4.9). *Similar to our previous examples,* $\alpha_1(t) = 0$ *and* $\alpha_2(t) = kt^2$. *The left side of* (4.10) *becomes*

$$\begin{aligned} \frac{2(b-a)^3}{3} \sum_{\ell=1}^{2} \int_a^b \alpha_\ell(s)\, ds \sum_{k=0}^{\ell-1} N^k &= \frac{2\pi^3}{3} \int_0^\pi ks^2\, ds(1+N) \\ &= \frac{2\pi^6}{9}(1+N)k. \end{aligned}$$

As defined in Theorem 4.6, $N = \frac{3|c-d|}{(b-a)} + K_3\left(\frac{17(b-a)^3}{12}\right)$. *In this case,* $N = \frac{1}{2} + \frac{17\pi^5}{12}k$. *Thus,*

$$\begin{aligned} \frac{2\pi^6}{9}(1+N)k &= \frac{2\pi^6}{9}\left(\frac{3}{2} + \frac{17\pi^5}{12}k\right)k \\ &= \frac{\pi^6}{3}k + \frac{17\pi^{11}}{54}k^2. \end{aligned}$$

Using the same methods as the previous two uniqueness examples, we find that,

$$k < \frac{3(\sqrt{102+9\pi} - 3\sqrt{\pi})}{17\pi^{\frac{11}{2}}} \approx 1.98346 \times 10^{-3}$$

for there to exist a unique solution to (4.8), (4.9).

Remark. The technique in this paper can not be used to show the existence or uniqueness of the boundary value problem

$$x^{(4)}(t) = f(t, x(t), x^{[2]}(t), ..., x^{[m]}(t)), \quad a < t < b, \tag{4.11}$$

$$x(a) = a\ ,\ x'(a) = 0\ ,\ x(b) = b,\ x'(b) = 0. \tag{4.12}$$

We note that as $c \to a$ or $d \to b$ in (1.4), then $\min\left\{\frac{12(b-d)}{7(b-a)^4}, \frac{12(c-a)}{7(b-a)^4}\right\} \to 0$. Consequently, $K_1 = 0$ and condition (H3) becomes $f \equiv 0$. The existence and uniqueness of solutions of (4.11), (4.12) remains an open problem.

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