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FINDING THE NATURAL SOLUTION TO $f(f(x)) = \exp(x)$

William Paulsen

Abstract. In this paper, we study the fractional iterates of the exponential function. This is an unresolved problem, not due to a lack of a known solution, but because there are an infinite number of solutions, and there is no agreement as to which solution is "best." We will approach the problem by first solving Abel's functional equation $\alpha(e^x) = \alpha(x)+1$ by perturbing the exponential function so as to produce a real fixed point, allowing a unique holomorphic solution. We then use this solution to find a solution to the unperturbed problem. However, this solution will depend on the way we first perturbed the exponential function. Thus, we then strive to remove the dependence of the perturbed function. Finally, we produce a solution that is in a sense more natural than other solutions.

1. Background

The problem of fractional iteration dates back to 1826 with Niels Abel [1], and expanded upon by Ernst Schröder [11] in 1871. In order to solve the equation $f(f(x)) = g(x)$ for general monotonically increasing functions $g(x)$, Abel considered the functional equation $\alpha(g(x)) = \alpha(x)+1$. If $\alpha(x)$ is a monotonically increasing solution to this equation, then $\alpha^{-1}(x)$ is well defined, so we can produce the function $f(x) = \alpha^{-1}(\alpha(x)+1/2)$. Then $f(f(x)) = \alpha^{-1}(\alpha(x) + 1) = g(x)$, so we have found the "half-iterate" of a function. Similarly, we can find fractional iterates of functions using the solution $\alpha(x)$. In fact, solving for α allows us to solve many functional equations involving $f(x)$. [8]

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The problem is that Abel's solution is far from unique. Not only can we add an arbitrary constant, but if $p(x)$ is any periodic function of period 1 such that $p'(x) > -1$, then $\alpha(x) + p(\alpha(x))$ will also be a solution to Abel's equation. (The condition $p'(x) > -1$ assures us that the new solution will also be monotonic.)

In order to get a unique solution, we have to consider functions $g(x)$ which have a *fixed point*. We say that x_0 is a fixed point of $g(x)$ if $g(x_0) = x_0$. Then if $s = g'(x_0) > 0$ and $g'(x_0) \neq 1$, we can solve the Schröder equation

$$\sigma(g(x)) = s\sigma(x).$$

In fact, if $g(x)$ is analytic at x_0, with a Taylor series of

$$g(x) = x_0 + s(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots,$$

then there will be a unique solution to the Schröder equation analytic at x_0 [6], normalized so that $\sigma'(x_0) = 1$. Its series is given by

$$\begin{aligned} \sigma(x) &= (x - x_0) - \frac{a_2}{s(s-1)}(x - x_0)^2 + \frac{2a_2^2 + (1-s)a_3}{s(s-1)(s^2-1)}(x - x_0)^3 \\ &+ \frac{(5s^3 - 3s^2 - 2s)a_2a_3 - (5s^2+1)a_2^3 + (s^3 - s^4 + s^2 - s)a_4}{s^2(s-1)(s^2-1)(s^3-1)}(x - x_0)^4 \\ &+ \cdots . \end{aligned} \tag{1}$$

Note that the nth term of the series for $\sigma(x)$ only depends on the first terms up to a_n of $g(x)$.

It is easy to convert a solution to Schröder's equation to a solution to Abel's equation. If we let

$$\alpha(x) = \frac{\ln(\sigma(x))}{\ln(s)},$$

then

$$\alpha(g(x)) = \frac{\ln(\sigma(g(x)))}{\ln(s)} = \frac{\ln(s\,\sigma(x))}{\ln(s)} = \frac{\ln(\sigma(x)) + \ln(s)}{\ln(s)} = \alpha(x) + 1.$$

If we use the normalized solution of Schröder's equation, we get

$$\begin{aligned} \alpha(x) &= \frac{\ln(x - x_0)}{\ln(s)} - \frac{a_2}{s(s-1)\ln(s)}(x - x_0) \\ &+ \frac{2a_3(s - s^2) + a_2^2(3s-1)}{2s^2(s-1)(s^2-1)\ln(s)}(x - x_0)^2 \\ &+ \left(\frac{a_2^3(4s^2 - 10s^3 + s - 1) + 3a_2a_3(4s^4 - 3s^3 - 2s^2 + s)}{3s^3(s-1)(s^2-1)(s^3-1)\ln(s)}\right. \end{aligned}$$

$$+ \frac{3a_4(s^4 - s^5 + s^3 - s^2)}{3s^3(s-1)(s^2-1)(s^3-1)\ln(s)}\Bigg) (x - x_0)^3$$
$$+ \quad \cdots .$$

Note that $\alpha(x)$ is discontinuous at the fixed point x_0, which is to be expected from observing Abel's equation. Yet the series will converge with a positive radius of convergence, so $\alpha(x) - \ln(x - x_0)/\ln(s)$ is analytic in a neighborhood of x_0. But there is a simple trick for analytically extending this function to a much larger region.

If $0 < g'(x_0) < 1$, then x_0 is called an *attractive* fixed point, since for points sufficiently close to x_0, $g(x_0)$ will be closer. The *basin of attraction* of x_0 is the set of points for which the sequence

$$\{x, g(x), g(g(x)), g(g(g(x))), \dots\}$$

converges to x_0. For any point in the basin of attraction, there is some iterate of x which is within the radius of convergence of Eq. 1. Then since

$$\sigma(g_n(x)) = s^n \sigma(x),$$

where $g_n(x)$ denotes the nth iterate of $g(x)$, and the left hand side is defined from Eq. 1 and is holomorphic, so is the right hand side. A similar approach can be used if $g'(x_0) > 1$, except we consider the set of points for which

$$\{x, g^{-1}(x), g^{-1}(g^{-1}(x)), g^{-1}(g^{-1}(g^{-1}(x))), \dots\}$$

converges to x_0 for an appropriately defined $g^{-1}(x)$. We can call this set the *basin of repulsion*.

If $g'(x_0) = 0$, the fixed point at x_0 is called *superattracting*. In particular, if

$$g(x) = x_0 + c(x - x_0)^m + \mathcal{O}((x - x_0)m + 1)$$

for $m \geq 2$, then we can find an analytic solution [10] to *Böttcher's equation*

$$\beta(g(x)) = (\beta(x))^m$$

in the neighborhood of the fixed point, with $\beta(x_0) = 0$. If we then let $\sigma(x) = \ln(\beta(x))$, then $\sigma(x)$ will solve Schröder's equation with $s = m$, although this introduces a logarithmic singularity at x_0. Finally, letting

$$\alpha(x) = \frac{\ln(\sigma(x))}{\ln s} = \frac{\ln(\ln(\beta(x)))}{\ln m}$$

would give us a solution to Abel's equation.

In particular, if $g(x)$ is a polynomial of degree $m > 1$, we can consider the point of ∞ to be a superattractive fixed point, using the transformation $x = 1/t$. If $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_m x^m$ for $m \geq 4$, then $\beta(x)$ can be given [2] by the Laurent series

$$\begin{aligned}\beta(x) = \sqrt[m-1]{a_m}\Big[x \quad &+ \quad \frac{a_{m-1}}{m a_m} + \frac{(1-m)a_{m-1}^2 + 2m a_m a_{m-2}}{2m^2 a_m^2 x} \\ &+ \quad \frac{(2m^2 - 3m + 1)a_{m-1}^3 + 6(m - m^2) a_m a_{m-1} a_{m-2}}{6m^3 a_m^3 x^2} \\ &+ \frac{6m^2 a_m^2 a_{m-3}}{6m^3 a_m^3 x^2} + \cdots\Big]. \end{aligned} \tag{2}$$

Schröder's equation can only be applied to a function with a fixed point, because the derivative at the fixed point is part of the equation. However, we can remove the dependence on s by considering the function

$$\lambda(x) = \frac{\sigma(x)}{\sigma'(x)\ln s}.$$

Then since

$$\sigma'(g(x))g'(x) = [\sigma(g(x))]' = [s\sigma(x)]' = s\sigma'(x),$$

$$\lambda(g(x)) = \frac{\sigma(g(x))}{\sigma'(g(x))\ln s} = \frac{s\sigma(x)g'(x)}{\sigma'(g(x))g'(x)\ln s} = \frac{s\sigma(x)}{s\sigma'(x)\ln s}g'(x) = \lambda(x)g'(x),$$

we find that $\lambda(x)$ solves *Julia's equation* $\lambda(g(x)) = \lambda(x)g'(x)$ [4]. At either an attractive or repulsive fixed point, this produces an analytic solution of Julia's equation in the neighborhood of the fixed point:

$$\begin{aligned}\lambda(x) \quad &= \quad \ln(s)\Big[(x - x_0) + \frac{a_2}{(s-1)s}(x - x_0)^2 + \frac{2(sa_3 - a_2^2)}{(s-1)s^2(s+1)}(x - x_0)^3 \\ &+ \quad \frac{(5s+4)a_2^3 - (8s^2 + 7s)a_2 a_3 + 3(s^3 + s^2)a_4}{(s-1)s^3(s+1)(s^2+s+1)}(x - x_0)^4 + \cdots\Big]. \end{aligned} \tag{3}$$

However, this is not the only analytic solution, since multiplying this solution by a constant yields another solution. It is trickier to normalize the solution to Julia's equation, since to reconstruct a solution of Abel's equation from $\lambda(x)$, we find that

$$\alpha(x) = C\int \frac{1}{\lambda(x)}\,dx$$

for some constant C. If we normalize the solution to Julia's equation so that

$$\int_x^{g(x)} \frac{1}{\lambda(x)}\, dx = 1$$

for all x, then letting $C = 1$ will allow $\alpha(g(x)) = \alpha(x) + 1$.

2. Previous Attempts

Finding the fractional iteration of e^x is related to the tetration problem, for which we define ${}^n a = a^{a^{\cdot^{\cdot^{a}}}}$, for which the a appears n times. To extend the idea of tetration for fractional n, we need to find the fractional iterates of $g(x) = a^x$, in particular, the inverse to Abel's function $\alpha(x)$. Although a^x has a real fixed point for $a \leq e^{1/e}$, there are no real fixed points for $a > e^{1/e}$. Hence, we cannot directly use the methods of Schröder and Abel to find fractional iterates of a^x. It is best if we first concentrate on the fractional iterates of e^x, with hopes of extending the ideas to other a^x later.

One of the earlier attempts to find a fractional interations of e^x is by Kneser [5], using the complex fixed points of e^x, in particular two approximated by $0.3181315052 \pm 1.3372357014i$. If we compute the holomorphic solutions to Schröder's equation centered at one of these fixed points, we can create a solution to Abel's equation for a portion of the complex plane, which unfortunately is not real on the real axis. However, Kneser was able to find a conformal mapping which converted this to a real analytic function that also solves Abel's equation. Because this solution utilizes the Riemann mapping theorem, it is extremely difficult to evaluate numerically. At least it does prove the existence of a real analytic solution. In [12], Kneser's solution was proven to be the unique solution that satisfies a certain uniqueness criterion.

The naïve approach is to assume that $\alpha(x)$ has a Maclaurin series of the form

$$\alpha(x) = \sum_{n=0}^{\infty} b_n x^n$$

and force the Maclaurin series for $\alpha(e^x) - \alpha(x) - 1$ to be term-wise equal to 0. We can assume that $b_0 = -1$ (so that $\alpha(1) = 0$), and we find the other coefficients satisfy

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 2 & 9 & 16 & 25 & \cdots \\ 1 & 8 & 21 & 64 & 125 & \cdots \\ 1 & 16 & 81 & 232 & 625 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

The general pattern for the matrix is $m_{i,j} = j^{i-1} - j!\delta_{i-1,j}$, where δ is the Kronecker delta. This gives us an infinite number of equations with an infinite number of unknowns, which generally has an infinite number of solutions. If we truncate the matrix to an n by n matrix, we will get n equations with n unknowns, which can be solved to produce an nth degree polynomial. But do these polynomials converge to a single function?

At first the polynomials seem to converge to a function

$$\begin{aligned} \alpha(x) \approx\ & -1 + 0.91594605x + 0.249354561x^2 - 0.1104647x^2 \\ & - 0.09393627x^3 + 0.0100031x^4 + 0.0358979x^5 \\ & + 0.0065736x^6 - 0.0123067x^7 - 0.00638988x^8 + 0.0032733x^9 \\ & + 0.0037691x^{10} + \cdots, \end{aligned} \tag{4}$$

but apparently the individual coefficients do not settle down beyond 7 or 8 places. Each polynomial solution gives an excellent approximation to a solution to Abel's equation, but the polynomials are not extremely close to each other. Actually, this is not surprising, since there are an infinite number of solutions to Abel's equation, so there is no reason for the polynomials to approach one particular solution.

We have a little better luck with Julia's equation. If we force the Maclaurin series

$$\lambda(x) = \sum_{n=0}^{\infty} c_n x^n$$

to satisfy $\lambda(e^x) = \lambda(x)e^x$, we find that, if we assume $c_0 = 1$, the other coefficients satisfy

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 3 & 4 & 5 & \cdots \\ -1 & 2 & 9 & 16 & 25 & \cdots \\ -2 & 2 & 21 & 64 & 125 & \cdots \\ -3 & 4 & 57 & 232 & 625 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix},$$

where this time the pattern for the matrix is $m_{i,j} = j^{i-1} - (i-1)!/(i-j-1)!$. This time the polynomials seem to converge to at least 16 places. We then find that

$$\begin{aligned} \lambda(x) \approx\ & c_0(1 - 0.544474419432800002x + 0.658257848047694222x^2 \\ -\ & 0.145172228840706x^3 + 0.0392401849104522x^4 \\ -\ & 0.0092762942201873x^5 + 0.00154819408904525x^6 \\ -\ & 0.0001236550531523x^7 + 0.00000869580079783x^8 \\ -\ & 0.0000114674312172x^9 + 0.00000225232705059x^{10} \\ +\ & 0.0000013297632729x^{11} - 0.0000002546678364x^{12} \\ -\ & 0.000000257792910611x^{13} + 0.0000000285251852x^{14} + \cdots) \end{aligned} \tag{5}$$

We can then numerically compute the value $c_0 \approx 1.0917673512583209922$ such that $\int_0^1 1/\lambda(x)\,dx = 1$.

Unfortunately, it would be very difficult to prove that as we let $n \to \infty$, the coefficients of the solutions converge to a single solution. Even if they did, there would be a limit to the precision we could obtain simply because of the complexity of solving n equations with n unknowns. So although this seems to give us a natural solution that we are looking for, it is impractical to use.

Another approach given in [13] is to consider the function $e^x - 1$, which does have a fixed point at $x = 0$. Unfortunately, this fixed point is *semi-stable*, since iterating negative values get closer to the fixed point, but iterating positive values get further away from the fixed point. Schröder's equation does not work for the case $s = 1$, but for Julia's equation, we can take the limit as $s \to 1$ of Eq. 3 to obtain

$$\lambda(x) = a_2(x-x_0)^2 + (a_3 - a_2^2)(x-x_0)^3 + \frac{3a_2^3 - 5a_2a_3 + 2a_4}{2}(x-x_0)^4 + \cdots.$$

We can then plug in the coefficients for $e^x - 1$ to produce

$$\lambda(x) \sim \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{48} - \frac{x^5}{180} + \frac{11x^6}{8640} - \frac{x^7}{6720} + \frac{11x^8}{241920} + \frac{29x^9}{1451520} - \frac{493x^{10}}{43545600} + \cdots$$

as $x \to 0^+$.

It should be noted that this series actually has a zero radius of convergence, hence it is written as an asymptotic series. Such series are not useless, since they can be converted to a form that is convergent. For example, we can convert the series into a convergent continued fraction

$$\lambda(x) = \frac{x^2}{2}\Big/\Big(1+\frac{x}{6}\Big/\Big(1+\frac{x}{12}\Big/\Big(1+\frac{x}{20}\Big/\Big(1-\frac{19x}{30}\Big/\Big(1+\frac{472x}{399}\Big/ \Big(1-\frac{8249x}{62776}\Big/\Big(1+\cdots.$$

We can compute $\psi(x) = \int 1/\lambda(x)\,dx$ to be

$$\psi(x) \sim -\frac{2}{x}+\frac{\ln(x)}{3}-\frac{x}{36}+\frac{x^2}{540}+\frac{x^3}{7776}-\frac{71x^4}{435456}+\frac{8759x^5}{163296000} + \frac{31x^6}{20995200}-\frac{183311x^7}{16460236800}+\cdots \quad \text{as } x\to 0^+, \tag{6}$$

which converted to a continued fraction is

$$\psi(x) = -\frac{2}{x}+\frac{\ln x}{3}-\frac{x}{36}\Big/\Big(1+\frac{x}{15}\Big/\Big(1-\frac{49x}{360}\Big/\Big(1+\frac{8425x}{12348}\Big/ \Big(1-\frac{163067111x}{693546000}\Big/\Big(1+\cdots. \tag{7}$$

Then for $x > 0$, $\psi(x)$ will satisfy

$$\psi(e^x - 1) = \psi(x) + 1.$$

It is known from [3] that since $\log(x + 1)$ is analytic in a neighborhood of 0, then there will be a unique solution ψ in a neighborhood of 0 that has the asymptotic relation in Eq. 6.

Of course this is not exactly the equation we are trying to solve. But we can argue that there is a unique $\alpha(x)$ solving $\alpha(e^x) = \alpha(x) + 1$ such that $\alpha(x) \sim \psi(x)$ as $x \to \infty$. It is easy to extend $\psi(x)$ analytically to the positive real axis, since this axis is in the basin of attraction of $\ln(x + 1)$, the inverse of $e^x - 1$.

Now, for a given x, we can compute $\alpha(x)$ by noting that

$$\alpha(x) = \alpha(e^x) - 1 = \alpha\left(e^{e^x}\right) - 2 = \alpha\left(e^{e^{e^x}}\right) - 3 = \cdots$$

If we express the iterated exponential function as $\exp^n(x)$, we have that

$$\alpha(x) = \alpha(\exp^n(x)) - n \quad \text{for all } n.$$

Since we are assuming that $\alpha(x) \sim \psi(x)$ as $x \to \infty$, we have that

$$\alpha(x) = \lim_{n\to\infty} \psi(\exp^n(x)) - n. \tag{8}$$

By defining $\alpha(x)$ in this way, we get a unique solution to Abel's equation for which $\alpha(x) \sim \psi(x)$ as $x \to \infty$. We also find that this α is real analytic for real x.

With this definition, we find that $\alpha(1) \approx -1.4419775343579015$. But we can add a constant to α so that $\alpha(1) = 0$. This way, the inverse function $\alpha^{-1}(x)$ solves the tetration problem. The graph is shown in Fig. 1.

Although this solution is often used, there are some drawbacks to this method. First of all, it is inconsistent with the "natural" solution given in Eq. 5. In this version, we find that $\alpha(1/2) - \alpha(1) \approx -0.498498375$, whereas using Eq. 8, we get $\alpha(1/2) - \alpha(1) \approx -0.497732466$. Also, it is time consuming to calculate $\psi(x)$ accurately. The main strategy for computing $\psi(x)$ is to repeatedly apply $\ln(x + 1)$ until the argument is close to 0, and then use the power series or continued fraction. However, it takes about 10000 iterations of $\ln(x + 1)$ to get the argument to less than 0.0001, because of the semi-attractive fixed point. Also, try this on other exponential functions, such as $a^x - 1$, we find that this the fixed point at 0 switches to an attractive fixed point when $a < e$, resulting in a discontinuity in the a variable if we compute the tetration.

Yet another solution is described in section 8 of [7]. If we inductively define

$$h_0(x) = x, \qquad h_n(x) = x + e^{h_{n-1}(x-1)} \quad \text{for } n > 0,$$

then the sequences $h_n(x)$ rapidly converges to a function $h_\infty(x)$, which has a growth similar to tetration. We then define

$$g_\infty(x) = \lim_{n\to\infty} \exp^{-n}(h_\infty(x + n)).$$

Finally, $\alpha_{g_\infty}(x) = g_\infty^{-1}(x)$ solves Abel's equation. Although calculating g_∞ is fairly easy, it is difficult to calculate the inverse function. Like the previous solution, we are using the "helper function" $h_\infty(x)$ which solves a similar functional equation to Abel's equation, and then use this to find a solution to Abel's equation with the same growth rate as $x \to \infty$.

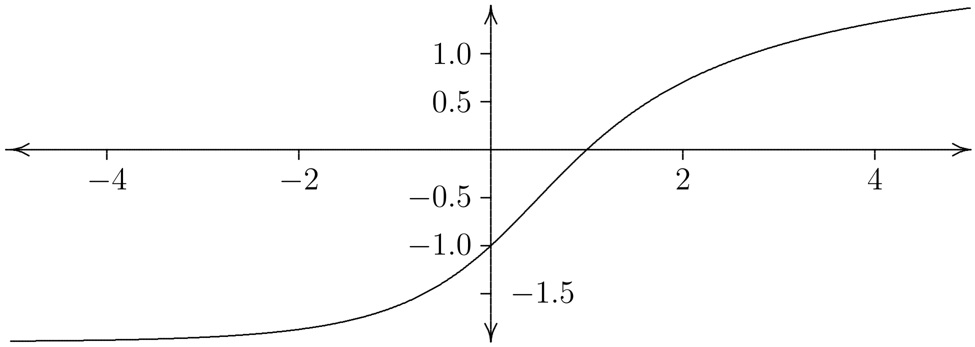

FIGURE 1. Graph of $\alpha(x)$, solving $\alpha(e^x) = \alpha(x) + 1$

3. Perturbing the problem

Our strategy is to solve a slightly different problem that is much easier. We consider adding a small perturbation $\epsilon(x)$ to the exponential function so that $e^x + \epsilon(x)$ will have a real fixed point. We require the following properties to hold for $\epsilon(x)$:

- $\epsilon(0) = -1$.
- $\epsilon(x)$ is analytic for all real numbers.
- $\epsilon'(x) > 0$ for all $x \geq 0$.
- $\lim\limits_{x\to\infty} \epsilon(x) = 0$.

Examples of such functions are $-e^{ax}$ and $-1/(x+1)^a$ for any positive constant a. These four properties will make $x = 0$ a fixed point for $g(x) = e^x + \epsilon(x)$, with $g'(0) > 1$. Also, $g(x)$ will be real analytic and monotonically increasing, and asymptotic to e^x as $x \to \infty$. These are precisely the properties needed to ensure that Schröder's functional equation for $g(x)$ has a unique normalized solution in the neighborhood of zero, and can be analytically extended to the positive real axis. If we convert this solution to a solution of Abel's equation, we can call this new function $\psi_\epsilon(x)$. Thus, we have

$$\psi_\epsilon(e^x + \epsilon(x)) = \psi_\epsilon(x) + 1, \qquad \psi_\epsilon(x) = \frac{\ln(x)}{\ln(1+\epsilon'(0))} + \mathcal{O}(x) \text{ as } x \to 0^+.$$

We can now use this function to solve the main equation, namely $\alpha(e^x) = \alpha(x) + 1$. If we assume that $\alpha(x) \sim \psi_\epsilon(x)$ as $x \to \infty$, we will produce the unique solution $\alpha_\epsilon(x)$, given by

$$\alpha_\epsilon(x) = \lim_{n\to\infty} \psi_\epsilon(\exp^n(x)) - n. \tag{9}$$

Furthermore, since $\psi_\epsilon(g^{-1}(x)) = \psi_\epsilon(x) - 1$, we also have that

$$\alpha_\epsilon(x) = \lim_{n\to\infty} \psi_\epsilon(g^{-n}(\exp^n(x))). \tag{10}$$

This will be a solution to Abel's equation, but it will depend upon the perturbation function $\epsilon(x)$. The goal is to eliminate this dependency. We will first demonstrate that the function $\alpha_\epsilon(x)$ only depends on the local behavior of $\epsilon(x)$ near the fixed point. To do this, we need a way to compare two solutions of Abel's equation.

We already observed that if $\alpha(x)$ is one solution to Abel's equation, then $\alpha(x) + p(\alpha(x))$ will also be a solution for a periodic function $p(x)$. Given two solutions to Abel's equation, α_{ϵ_1} and α_{ϵ_2}, we can compute the periodic function relating the two by

$$p(x) = \alpha_{\epsilon_1}(\alpha_{\epsilon_2}^{-1}(x)) - x.$$

For example, if $\epsilon_1(x) = e^{-x}$ and $\epsilon_2(x) = e^{-2x}$, we produce the periodic function shown in Fig. 2.

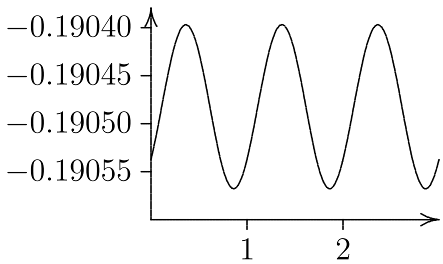

FIGURE 2. $\alpha_{\epsilon_1}(\alpha_{\epsilon_2}^{-1}(x)) - x$ is periodic.

The difference between the peaks and the troughs is small in Fig. 2, about 0.000171. This gives us a way to measure the difference in the solutions of $\alpha_{\epsilon_1}(x)$ and $\alpha_{\epsilon_2}(x)$. However, this method will give a slightly different result if we interchange the roles of ϵ_1 and ϵ_2. Rather, we can use the fact that since $p(x)$ is periodic, then so is

$$\ln(p'(x) + 1) = \ln\left(\frac{\alpha'_{\epsilon_1}(\alpha_{\epsilon_2}^{-1})}{\alpha'_{\epsilon_2}(\alpha_{\epsilon_2}^{-1})}\right).$$

Now interchanging $\alpha_{\epsilon_1}(x)$ and $\alpha_{\epsilon_2}(x)$ will negate this expression, plus evaluate it at $\alpha_{\epsilon_2}(\alpha_{\epsilon_1}^{-1}(x))$. Thus, the difference between peaks and troughs of $\ln(p'(x)+1)$ is independent of the order of ϵ_1 and ϵ_2. If we let this difference be $d(\alpha_{\epsilon_1}, \alpha_{\epsilon_2})$, then we have a metric on the set of normalized C_1 solutions to Abel's equation. (The triangle inequality is an easy consequence of the variance of the sum of two periodic functions can be no more than the sum of the individual variances.) We can refer to this as the *log ratio metric*. In the example above, $d(\alpha_{\epsilon_1}, \alpha_{\epsilon_2}) \approx 0.0010724$. Note that the solution given in [7] does not have a continuous derivative, so this metric could not be used for this solution.

4. Exploring the Local Behavior

In order to show that the effect of perturbing the problem by $\epsilon(x)$ only depends on the local behavior of ϵ, we can try the perturbation trick on a polynomial. Let T_m be the m-th degree Taylor polynomial of e^x centered at 0,

$$T_m = \sum_{i=0}^{m} \frac{x^i}{i!}.$$

The goal is to solve Abel's equation for T_m for $m \geq 1$ instead of e^x. That is, we will find a solution to

$$A(T_m) = A(x) + 1.$$

We will again perturb this by the function $\epsilon(x)$, where $\epsilon(0) = -1$, $\epsilon'(x) > 0$, and $\lim_{x\to\infty} \epsilon(x) = 0$. Then $G(x) = T_m(x) + \epsilon(x)$ will have a repulsive fixed point at $x = 0$, so we can find a unique normalized solution to Schröder's equation analytically extended to include the positive real axis. By converting this to a solution to Abel's equation, we get the function $\Psi_{m,\epsilon}$ for which

$$\Psi_{m,\epsilon}(T_m+\epsilon(x)) = \Psi_{m,\epsilon}(x)+1, \quad \Psi_{m,\epsilon}(x) = \frac{\ln(x)}{\ln(1+\epsilon'(0))} + \mathcal{O}(x) \text{ as } x \to 0^+.$$

From this solution, we can find a solution for Abel's equation over the Taylor polynomial T_m. We find a solution that is asymptotic to $\Psi_{m,\epsilon}$ as $x \to \infty$. If we let $T_m^n(x)$ be the n-th iterate of the m-th Taylor polynomial, then we can let

(11) $$A_{m,\epsilon}(x) = \lim_{n\to\infty} \Psi_{m,\epsilon}(T_m^n(x)) - n.$$

Once again, if $\epsilon_1(x)$ and $\epsilon_2(x)$ are two different perturbation functions, then for each m, $A_{m,\epsilon_1}(A_{m,\epsilon_2}^{-1}(x)) - x$ will be a periodic function. We can show that these periodic functions converge. First we need to prove a simple lemma.

LEMMA 1. *Let $g_1(x)$ and $g_2(x)$ be two increasing functions, for which $g_1(x) > x$, $g_2(x) > x$ and $[g_1^{-1}]' < 1/2$ for $x > N - 1$. Let $0 < \delta < 1$ be such that $|g_1(x) - g_2(x)| < \delta$ for $x > N$, and let $f(x)$ be a function for which $|f(x) - x| < \delta$ for $x > N$. Then*

$$|g_1^{-1}(f(g_2(x))) - x| < \delta \qquad \textit{for} \quad x > N.$$

Proof. If $x > N$, $g_2(x) > N$, so $|f(g_2(x)) - g_2(x)| < \delta$, hence $|f(g_2(x)) - g_1(x)| < 2\delta$. Since $[g_1^{-1}]' < 1/2$, we have by the mean value theorem that

$$\left| \frac{g_1^{-1}(a) - g_1^{-1}(b)}{a - b} \right| < \frac{1}{2} \qquad \text{whenever} \quad a, b > N - 1, \quad a \neq b.$$

In particular, $g_1(x) > N$ and $f(g_2(x)) > N - \delta > N - 1$ when $x > N$, so

$$|g_1^{-1}(f(g_2(x))) - g_1^{-1}(g_1(x))| < \frac{1}{2}|f(g_2(x)) - g_1(x)| < \delta \qquad \text{for} \quad x > N.$$

□

PROPOSITION 1. *Let $\epsilon_1(x)$ and $\epsilon_2(x)$ be two analytic increasing functions with $\epsilon_1(0) = \epsilon_2(0) = -1$, and $\epsilon_1(x) \to 0$ and $\epsilon_2(x) \to 0$ as $x \to \infty$. Then the sequence of periodic functions $A_{m,\epsilon_1}(A_{m,\epsilon_2}^{-1}(x)) - x$ converge uniformly to $\alpha_{\epsilon_1}(\alpha_{\epsilon_2}^{-1}(x)) - x$.*

Proof. Let $0 < \delta < 1$. We need to show that for sufficiently large m,

$$|\alpha_{\epsilon_1}(\alpha_{\epsilon_2}^{-1}(x)) - A_{m,\epsilon_1}(A_{m,\epsilon_2}^{-1}(x))| < \delta$$

for all x. This is equivalent to saying that

$$|\alpha_{\epsilon_1}(\alpha_{\epsilon_2}^{-1}(A_{m,\epsilon_2}(A_{m,\epsilon_1}^{-1}(x)))) - x| < \delta.$$

Because both $\epsilon_1(x)$ and $\epsilon_2(x)$ approach 0 as $x \to \infty$, as well as $\psi'_{\epsilon_1}(x)$, there is an $N > 3$ such that

(12) $$|\epsilon_1(x)| < \frac{\delta}{4}, \quad |\epsilon_2(x)| < \frac{\delta}{4} \quad \text{and} \quad \psi'_{\epsilon_1}(x) < 1 \quad \text{for all} \quad x > N.$$

Although Ψ_{m,ϵ_1} and ψ_{ϵ_1} are not analytic at $x = 0$, the combination $\Psi^{-1}_{m,\epsilon_1}(\psi_{\epsilon_1}(x)) = S^{-1}_{m,\epsilon_1}(\sigma_{\epsilon_1}(x))$, where S_{m,ϵ_1} and $\sigma_{\epsilon_1}(x)$ are the unique normalized analytic solution to Schröder's equation for $T_m + \epsilon_1(x)$ and $e^x + \epsilon_1(x)$, respectively. Since the first m coefficients of S_{m,ϵ_1} and S_{m,ϵ_2} will match the first m coefficients of σ_{ϵ_1} and σ_{ϵ_2} respectively, there is some M such that

$$(13)\qquad |\Psi^{-1}_{m,\epsilon_1}(\psi_{\epsilon_1}(x)) - x| < \frac{\delta}{4} \qquad \text{and} \qquad |\psi^{-1}_{\epsilon_2}(\Psi_{m,\epsilon_2}(x)) - x| < \frac{\delta}{4}$$

for all $m \geq M$ on the finite interval $0 < x < e^{N+1} + 1$. Note that although Eq.12 is valid for the outer region $x > N$, and Eq. 13 is valid in the inner region $x < e^{N+1} + 1$, there is an overlap region in which both estimates are valid.

Let $g_1(x) = e^x + \epsilon_1(x)$, and $g_2(x) = e^x + \epsilon_2(x)$. It is clear that $f(x) = x$ satisfies $|f(x) - x| < \delta/4$, so by induction, we can use Lemma 1 to show that

$$|g_1^{-n}(g_2^n(x)) - x| < \frac{\delta}{4} \qquad \text{for all} \quad n \quad \text{when} \quad x > N.$$

Since we can express

$$\alpha_{\epsilon_1} = \lim_{n\to\infty} \psi_{\epsilon_1}(g_1^{-n}(\exp^n(x))), \qquad \alpha^{-1}_{\epsilon_2} = \lim_{n\to\infty} \exp^{-n}(g_2^n(\psi^{-1}_{\epsilon_2}(x))),$$

we have that

$$\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(\psi_{\epsilon_2}(x)))) = \lim_{n\to\infty} g_1^{-n}(g_2^n(x)).$$

Therefore,

$$|\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(\psi_{\epsilon_2}(x)))) - x| < \frac{\delta}{4} \quad \text{when} \quad x > N.$$

Likewise, for any $m \geq M$, we find that T_m satisfies the conditions of Lemma 1 for $N > 3$, so

$$|\Psi^{-1}_{m,\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\Psi_{m,\epsilon_1}(x)))) - x| < \frac{\delta}{4} \quad \text{when} \quad x > N.$$

If $N + \delta/4 < x < e^{N+1} + 1 - \delta/4$, then $N < \psi^{-1}_{\epsilon_2}(\Psi_{\epsilon_1}(x)) < e^{N+1} + 1$, so

$$|\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(\psi_{\epsilon_2}(\psi^{-1}_{\epsilon_2}(\Psi_{m,\epsilon_2}(x)))))) - \psi^{-1}_{\epsilon_2}(\Psi_{m,\epsilon_2}(x))| < \frac{\delta}{4}.$$

Hence, we have

$$|\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(\Psi_{m,\epsilon_2}(x)))) - x| < \frac{\delta}{2}.$$

If $N + \delta/2 < x < e^{N+1} + 1 - \delta/2$, then $N + \delta/4 < \Psi^{-1}_{m,\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\Psi_{m,\epsilon_1}(x)))) < e^{N+1} + 1 - \delta/4$, so

$$\begin{aligned}|\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(\Psi_{m,\epsilon_2}(\Psi^{-1}_{m,\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\Psi_{m,\epsilon_1}(x)))))))) \\ -\Psi^{-1}_{m,\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\Psi_{m,\epsilon_1}(x))))| < \frac{\delta}{2},\end{aligned}$$

hence

$$|\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\Psi_{m,\epsilon_1}(x)))))) - x| < \frac{3\delta}{4}.$$

Finally, if $N + 3\delta/4 < x < e^{N+1} + 1 - 3\delta/4$, then $N + \delta/2 < \Psi^{-1}_{m,\epsilon_1}(\psi_{\epsilon_1}(x)) < e^{N+1} + 1 - \delta/2$, so

$$|\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\Psi_{m,\epsilon_1}(\Psi^{-1}_{m,\epsilon_1}(\psi_{\epsilon_1}(x))))))))-\Psi^{-1}_{m,\epsilon_1}(\psi_{\epsilon_1}(x))| < \frac{3\delta}{4}.$$

From this, we have that

$$|\psi^{-1}_{\epsilon_1}(\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\psi_{\epsilon_1}(x)))))) - x| < \delta.$$

In particular, since $\delta < 1$, this will be valid on the interval $N+1 \leq x \leq e^{N+1}$.

Since $\psi'_{\epsilon_1}(x) < 1$ for $x > N + 1$, we can use the mean value theorem to show that

$$|\psi_{\epsilon_1}(a) - \psi_{\epsilon_1}(b)| < |a - b|$$

when both a and b are at least $N + 1$. So

$$|\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(\psi_{\epsilon_1}(x))))) - \psi_{\epsilon_1}(x)| < \delta$$

for $N + 1 \leq x \leq e^{N+1}$, so

$$|\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(x)))) - x| < \delta$$

for $\psi^{-1}_{\epsilon_1}(N+1) \leq x \leq \psi^{-1}_{\epsilon_1}(e^{N+1})$. Since $e^{N+1} + \epsilon_1(N+1) < e^{N+1}$, we see that this interval is over one unit in length. Since $\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(x)))) - x$ is periodic with period 1, this means that

$$|\alpha_{\epsilon_1}(\alpha^{-1}_{\epsilon_2}(A_{m,\epsilon_2}(A^{-1}_{m,\epsilon_1}(x)))) - x| < \delta$$

for all x. $\square$

5. Comparing with the Natural Solution

At this point, we can compare any two solutions of Abel's equation $\alpha(e^x) = \alpha(x) + 1$. If there were a "natural" solution to Abel's equation, we could compare this solution with another solution produced by the perturbation $\epsilon(x)$ by the periodic function $p(x) = \alpha(\alpha_\epsilon^{-1}((x)) - x$. In fact, if we know this periodic function, we can reproduce the natural solution $\alpha(x)$.

Proposition 1 shows that $p(x)$ would only depend on the local behavior of $\epsilon(x)$ and e^x, since replacing e^x with a Taylor polynomial approximation T_m produces approximately the same periodic function. But Abel's equation for T_m *does* have a natural solution, since T_m is a polynomial! We can solve Böttcher's equation $\beta_m(T_m(x)) = (\beta_m(x))^m$, and let $\alpha_m(x) = \ln(\ln(\beta_m(x)))/\ln(m)$. If indeed $\alpha_m(\alpha_\epsilon^{-1}((x)) - x$ converges to a periodic function as $m \to \infty$, we could use this period function to peel off the effects of the perturbation function $\epsilon(x)$, giving us a solution that is independent of which $\epsilon(x)$ that we chose.

Plugging a Taylor polynomial for e^x into Eq. 2 causes it to simplify greatly.

$$\beta_m(x) = (m!)^{1/(1-m)} \left(x + 1 + \frac{m-1}{2x} + \frac{(m-1)(2m-7)}{6x^2} \right. \\ \left. + \frac{(m-1)(6m^2 - 55m + 95)}{24x^3} + \cdots \right).$$

Since the point at ∞ will attract any positive number as long as $m > 1$, we can analytically extend this function to include the positive real axis. We can then let $A_m = \ln(\ln(\beta_m(x)))/\ln(m)$, which will satisfy $A_m(T_m) = A_m(x) + 1$.

For a given perturbation $\epsilon(x)$, we can see the effect of using the perturbed Taylor polynomial verses the natural Taylor polynomial by computing

$$p_{m,\epsilon}(x) = A_m(A_{m,\epsilon}^{-1}(x)) - x,$$

which will again be a periodic function. The natural question is whether $p_{m,\epsilon}(x)$ converges to a single periodic function $P_\epsilon(x)$ as $m \to \infty$. If it did, we could define $\alpha(x)$ to be

$$\alpha(x) = P_\epsilon(\psi_\epsilon) + \psi_\epsilon.$$

By proposition 1, this would not depend on which perturbation function ϵ we chose, so we would succeed in finding a natural solution to Abel's equation.

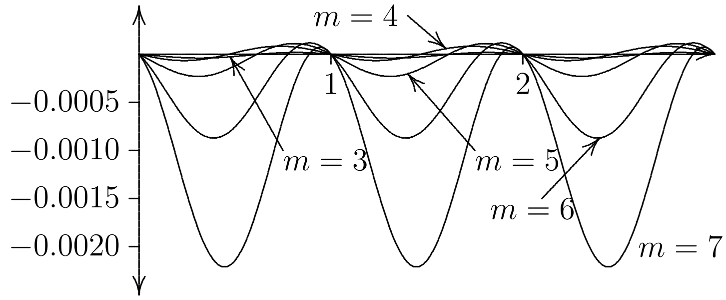

FIGURE 3. $A_m(A_{m,\epsilon}^{-1}(x)) - x$ for $m = 3, 4, 5, 6, 7$.

Unfortunately, the $p_{m,\epsilon}$ do not converge, even if first we normalize A_m and $A_{m,\epsilon}$ so that $A_m(1) = A_{m,\epsilon}(1) = 0$. Fig. 3 shows $p_{m,\epsilon}$ for $\epsilon(x) = -e^{-x}$ and $3 \le m \le 7$. The amplitude of the periodic functions grow exponentially with m, and this pattern persists with larger m. Thus, we cannot use the $p_{m,\epsilon}$ to remove the dependence of the $\epsilon(x)$ as we had hoped.

None-the-less, we can use the log ratio metric to find which $\epsilon(x)$ causes α_ϵ to come closest to the "natural solution" found in Eq. 4. The results are given in Table 1. We see that in fact using $\epsilon(x) = -e^{-2x}$ gives a very close approximation to the solution found in Eq. 4, even though the later is not a precise solution.

	$\epsilon = -1$	$\epsilon = -e^{-x}$	$\epsilon = -e^{-2x}$	$\epsilon = -e^{-3x}$	$g_\infty^{-1}(x)$
"Natural" $\alpha(x)$	0.0049983	0.0013073	0.0003448	0.0048961	0.0134455
$\epsilon = -1$		0.0062916	0.0052950	0.0082153	0.0183861
$\epsilon = -e^{-x}$			0.0010724	0.0047838	0.0121379
$\epsilon = -e^{-2x}$				0.0046229	0.0131907
$\epsilon = -e^{-3x}$					0.0133121

TABLE 1. Metric distance between various solutions.

Another question is whether there is an $\epsilon(x)$ for which the solution to Abel's equation is particularly easy to find. That is, we would like an $\epsilon(x)$ that satisfies the properties of Sect. 3, as well as

- $g(x) = e^x + \epsilon(x)$ is easy to invert.
- The entire half plane $\Im(x) > 0$ is in the basin of attraction for the fixed point 0 of $g^{-1}(x)$.

An obvious such $\epsilon(x)$ would be $\epsilon(x) = -e^{-x}$, since $e^x + \epsilon(x) = 2\sinh(x)$. This has an easy to compute inverse, $\sinh^{-1}(x/2)$. If we use the principle inverse, the entire complex plane is in the basin of attraction of the fixed point at 0. [9] In fact, calculating $\alpha_\epsilon(x)$ is possible on a scientific calculator. The fact that $e^x + \epsilon(x)$ is an odd function makes it particularly easy, since using only 2 terms of the series for $\psi(x)$ give exceptional accuracy.

1. Do $e^x \to x$ until $x > 230.25$, counting the number of times the operation is done.
2. Do $\sinh^{-1}(x/2) \to x$ the number of times in step 1. If the calculator doesn't have hyperbolic functions, use $\ln((x + \sqrt{x^2+1})/2) \to x$.
3. Do $\sinh^{-1}(x/2) \to x$ until $x < 0.01$, counting the number of times the operation is done.
4. Calculate $(\ln(x) - x^2/18)/\ln(2)$.
5. Add the count from step 3.
6. Subtract 0.06783836607 to normalize the function so that $\alpha(1) = 0$.

We can invert the function in a similar way , for $x > -2$.

1. Add 0.06783836607 so that $\alpha^{-1}(0) = 1$.
2. Subtract an integer so that $x < -7$.
3. Calculate $2^x + 8^x/18$.
4. Do $2\sinh(x) \to x$ the number of times in step 2. Note that $2\sinh(x) = e^x - e^{-x}$.
5. Do $2\sinh(x) \to x$ until $x > 230.25$, counting the number of times.
6. Do $\ln x \to x$ the number of times in step 5.

These routines are guaranteed to give 10 places of accuracy, and are designed to balance accumulative error and limit errors. Since $f(x) = \alpha^{-1}(\alpha(x) + 1/2)$ solves the equation $f(f(x)) = e^x$, we have a way to compute the title problem.

6. Generalizations

Having found a plausible orthodox solution to Abel's equation for e^x, let us consider a way to generalize it to cover fractional iterations of a^x for $a > e^{1/e}$. This would solve the tetration problem. Note that when

$a = e^{1/e}$, a^x has a fixed point at $x = e$, so there is an orthodox tetration for $1lea \leq e^{1/e}$ using Julia's equation [14].

The obvious generalization would suggest letting $\epsilon(x) = -a^{-x}$, but $a^x - a^{-x}$ produces an additional positive fixed point for $a < \sqrt{e}$. In fact, when a is close to $e^{1/e}$, perturbing the function by any amount will cause a fixed point to be produced near $x = e$. In order to assure that there will be only one fixed point of the perturbed function for all $a > e^{1/e}$, we must have the fixed point move, depending on the value of a.

Let us determine where the function a^x comes closest to the line $y = x$, by finding the critical point of $a^x - x$. We find that this point is at $\xi_a = -\ln(\ln a)/\ln(a)$, so we will choose $\epsilon_a(x) = -ca^{-x}$ so that $a^x + \epsilon_a(x)$ will have a fixed point at ξ_a. This produces $c = (1 + \ln(\ln(a)))/\ln(a)^2$, so we find that

$$g_a(x) = a^x + \epsilon_a(x) = a^x - \frac{1 + \ln(\ln(a))}{(\ln(a))^2} a^{-x}.$$

We find that this function can still be readily inverted via the quadratic equation.

$$\begin{aligned} g_a^{-1}(x) = \quad & \frac{\ln\left(\frac{1}{2}\left(\sqrt{x^2(\ln(a))^2 - 2x\ln(\ln(a))\ln(a) + (\ln(\ln(a)) + 2)^2}\right)\right)}{\ln(a)} \\ & + \frac{\ln\left(\frac{1}{2}\left(x\ln(a) - \ln(\ln(a))\right)\right)}{\ln(a)}. \end{aligned}$$

Note that when $a = e$, $\epsilon_a(x)$ simplifies to $-e^{-x}$, and when $a = e^{1/e}$, $\epsilon_a = 0$. Thus we have a smooth transition between our solution for $a = e$ and the point where there is a known solution.

We can solve Abel's equation for $a^x + \epsilon_a(x)$, since there is now a fixed point at ξ_a. This produces the function $\psi_a(x)$, whose first three terms of the series expansion about $x = \xi_a$ are

$$\begin{aligned} \psi_a(x) \quad = \quad & \frac{1}{\ln(2 + \ln(\ln(a)))}\Big(\ln(x - \xi_a) \\ & + \frac{\ln(a)\ln(\ln(a))}{2(1 + \ln(\ln(a)))(2 + \ln(\ln(a)))}(x - \xi_a) \\ & + \frac{\ln(a)^2\left(5\ln(\ln(a))^3 - 5\ln(\ln(a))^2 - 32\ln(\ln(a)) - 16\right)}{24(\ln(\ln(a)) + 1)^2(\ln(\ln(a)) + 2)^2(\ln(\ln(a)) + 3)}(x - \xi_a)^2 \\ & + \cdots\Big). \end{aligned}$$

Then ψ_a solves the equation $\psi_a(a^x + \epsilon_a(x)) = \psi_a(x) + 1$. We can also find the series for the inverse function.

$$\begin{aligned}\psi_a^{-1}(x) &= \xi_a + (2+\ln(\ln(a)))^x - \frac{\ln(a)\ln(\ln(a))(2+\ln(\ln(a)))^{2x}}{2(1+\ln(\ln(a)))(2+\ln(\ln(x)))} \\ &+ \frac{(\ln(a))^2(\ln(\ln(a))^3 + 8\ln(\ln(a))^2 + 8\ln(\ln(a)) + 4)(2+\ln(\ln(a)))^{3x}}{6(1+\ln(\ln(a)))^2(2+\ln(\ln(a)))^2(3+\ln(\ln(a)))} \\ &+ \cdots .\end{aligned}$$

Finally, we can express

$$\alpha_a(x) = \lim_{n\to\infty} \psi_a(g_a^{-n}(\exp_a^n(x))), \quad \text{and} \quad \alpha_a^{-1}(x) = \lim_{n\to\infty} \exp_a^{-n}(g_a^n(\psi_a^{-1}(x))).$$

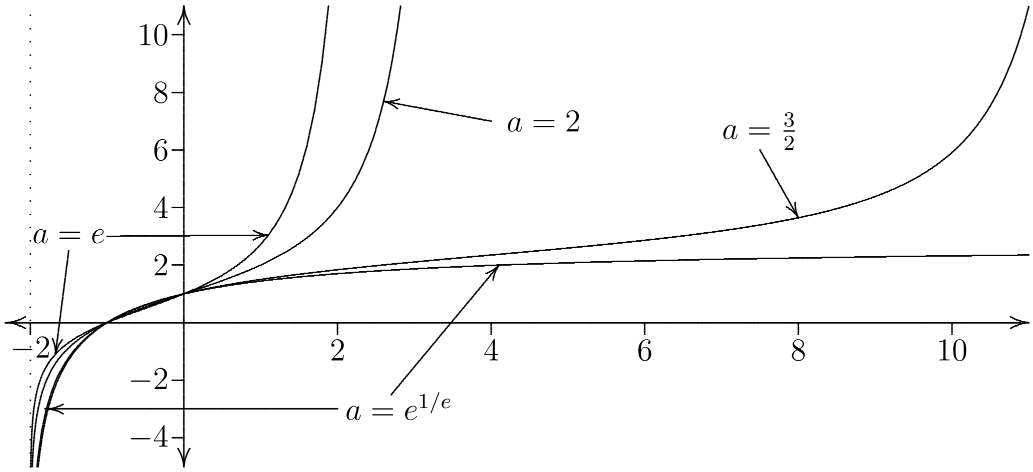

FIGURE 4. $\alpha_a^{-1}(x)$ for $a = e^{1/e}, 3/2, 2$, and e.

The graphs for $\alpha_a^{-1}(x)$ for various a are shown in Fig. 4. Also shown in this figure is $\alpha_{e^{1/e}}(x)$, which has a natural solution. Since

$$h\left((e^{1/e})^{h^{-1}(x)}\right) = e^x - 1 \qquad \text{when} \quad h = \ln x - 1,$$

we see that $(e^{1/e})^x$ is naturally conjugate to $e^x - 1$, which we have seen has a natural solution to Abel's equation in Eq. 7. Hence, we have

$$\alpha_{e^{1/e}}(x) = \psi(\ln x - 1) + C \tag{14}$$

for some constant C, using the ψ from Eq. 7.

We can see in Fig. 4 that as a approaches $e^{1/e}$, there is a section of the graph of $\alpha_a^{-1}(x)$ that is nearly horizontal, caused by the "bottleneck" created where the graph of $y = a^x$ is very close to the line $y = x$. This begs the question as to whether the limit of $\alpha_a^{-1}(x)$ approaches $\alpha_{e^{1/e}}^{-1}(x)$, at least pointwise, as $a \to e^{1/e}$. Proposition 1 suggests that the effect of adding $\epsilon(x)$ to solve Abel's equation only effects the solution based on the local behavior of $\epsilon(x)$, and since $\epsilon_a(x) \to 0$ as $a \to e^{1/e}$, this effect should vanish. However, the situation is different here than for proposition 1, so we need another proposition. First let us develop a useful lemma.

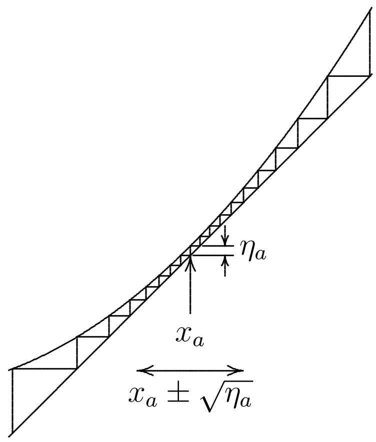

FIGURE 5. Analysis of iterations near a "bottleneck."

Lemma 2:

For each $a > a_0$, let $x < f_a(x)$ and $x < g_a(x)$ be increasing, concave up analytic functions on the interval $0 \le x \le N$. Suppose also that $f_a(x)$ and $g_a(x)$ both approach the analytic function $f_0(x)$ uniformly as $a \to a_0$, where $f_0(x)$ has a fixed point at x_0. Let x_a be the local minimum of $f_a(x) - x$, and ξ_a be the local minimum of $g_a(x) - x$. Suppose that

$$f_a(x_a) - x_a = g_a(\xi_a) - \xi_a = \eta_a,$$

that is, f_a and g_a have the same minimum distance to the line $y = x$. Let $h_a(x)$ be functions that uniformly converge to the function x. Finally, suppose that there is some M such that

$$\left| \frac{1}{\sqrt{f_a''(x_a)}} - \frac{1}{\sqrt{g_a''(\xi_a)}} \right| \le M\eta_a. \tag{15}$$

Then for all $\delta > 0$, there is an a_δ such that

$$|f_a^{-m}(h_a(g_a^m(x))) - x| < \delta \qquad \text{whenever} \quad a_0 < a < a_\delta \quad \text{and} \quad g_a^m(x) < N.$$

Proof: If there were some m such that $g_a^m(x) > N$ for all a, the proof would be obvious, since $g_a^m(x)$ and $f_a^m(x)$ would both uniformly converge to $f_0^m(x)$ as $a \to a_0$, and $h_a(x)$ is already uniformly converging to x. The issue is that if $0 \le x_0 \le N$, then as $a \to a_0$, the number of iterates of $f_a(x)$ and $g_a(x)$ needed to get past x_0 goes to infinity. See Fig. 5.

The region of interest is the "bottleneck" area, where $|x - x_a| = O(\sqrt{\eta_a})$. If $f_a''(x_a) = k_a$, we can approximate f_a within this region by

$$f_a(x) = \eta_a + x + k_a \frac{(x - x_a)^2}{2} + O(\eta_a^{3/2}).$$

If we divide the bottleneck region into subintervals of width Δx_i, then the number of iterates of $f_a(x)$ needed to get past this subinterval is approximately

$$\frac{\Delta x_i}{f_a(x_i^*) - x_i^*} \approx \frac{\Delta x_i}{\eta_a + k_a(x_i^* - x_a)^2/2},$$

where x_i^* is a representative point of the interval. The sum of these is a Riemann sum for the integral

$$\int_{-\infty}^{\infty} \frac{dx}{\eta_a + k_a(x - x_a)^2/2} = \left. \frac{\sqrt{2}\tan^{-1}(\sqrt{k_a}x/\sqrt{2\eta_a})}{\sqrt{k_a \eta_a}} \right|_{-\infty}^{\infty} = \frac{\pi\sqrt{2}}{\sqrt{\eta_a f_a''(x_a)}}.$$

Likewise, the number of iterates of $g_a(x)$ needed to get past the same region is $\pi\sqrt{2}/\sqrt{\eta_a g_a''(\xi_a)}$. Using Eq. 15, we see that the difference in the number of iterates is bounded by $\pi M \sqrt{2\eta_a}$, which goes to 0 as $a \to a_0$. If we had kept higher order terms, the corrections would also go to 0 as $a \to a_0$. Thus, $|f_a^{-m}(g_a^m(x)) - x| < \delta$ for a sufficiently close to a_0 if the iterates are within the bottleneck region. We have already covered the case outside this region, so the proof is complete. $\square$

The key to this lemma was the fact that f_a and g_a had very similar "bottlenecks." We can create a function with a similar bottleneck of a^x by defining

$$k_a(x) = e^{x/e} + \frac{1 + \ln(\ln(a)}{\ln(a)}.$$

We have already observed that the local minimum of $a^x - x$ is at $(-\ln(\ln(a)) / \ln(a), (1 + \ln(\ln(a))/\ln(a))$, and we designed $k_a(x) - x$ to have a local minimum at $(e, (1 + \ln(\ln(a))/\ln(a))$. Then since

$$\left|\frac{1}{\sqrt{f_a''(x_a)}} - \frac{1}{\sqrt{g_a''(\xi_a)}}\right| = \left|\frac{1}{\sqrt{\ln a}} - \sqrt{e}\right| < \frac{1 + \ln(\ln a)}{\ln a} \qquad \text{for} \quad a < 3,$$

we can use $M = 1$ in the conditions of the lemma.

Let us find a solution to Abel's equation for the function $k_a(x)$, valid for $x < e$. We already have a solution to $\alpha(e^{x/e}) = \alpha(x) + 1$ valid for $x > e$, namely $\alpha_{e^{1/e}}$ from Eq. 14. We can let

$$\phi_a(x) = \lim_{n\to\infty} \alpha_{e^{1/e}}(\exp_{e^{1/e}}^{-n}(k_a^n(x))).$$

Then even for $x < e$,

$$\begin{aligned}
\phi_a(k_a(x)) &= \lim_{n\to\infty} \alpha_{e^{1/e}}(\exp_{e^{1/e}}^{-n}(k_a^{n+1}(x))) \\
&= \lim_{n\to\infty} \alpha_{e^{1/e}}(\exp_{e^{1/e}}(\exp_{e^{1/e}}^{-n-1}(k_a^{n+1}(x))) \\
&= \lim_{n\to\infty} \alpha_{e^{1/e}}(\exp_{e^{1/e}}(\exp_{e^{1/e}}^{-n-1}(k_a^{n+1}(x)) + 1 = \phi_a(x) + 1.
\end{aligned}$$

PROPOSITION 2. *Given $\delta > 0$, there is an a_0 such that*

$$|\phi_a^{-1}(\alpha_a(x)) - x| < \delta \qquad \textit{for } x > 0 \textit{ and } e^{1/e} < a < a_0.$$

Proof. We can express

$$\phi_a^{-1}(x) = \lim_{n\to\infty} k_a^{-n}(\exp_{e^{1/e}}^n(\alpha_{e^{1/e}}^{-1}(x))).$$

Thus,

$$\phi_a^{-1}(\alpha_a(x)) = \lim_{n\to\infty} k_a^{-n}(\exp_{e^{1/e}}^n(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-n}(\exp_a^n(x)))))).$$

Since we can find an a_0 such that $|g_a(x) - a^x| < \delta/4$ for $e^{1/e} < a < a_0$, we can use Lemma 1 to show that

$$|g_a^{-n}(\exp_a^n(x)) - x| < \frac{\delta}{4}.$$

Likewise, we can pick a_0 small enough so that

$$|k_a^{-n}(\exp_{e^{1/e}}^n(x)) - x| < \frac{\delta}{4}.$$

However, the conditions of this lemma are only valid for $x > 2e + 1$. If it takes m iterations of a^x or k_a to get from x past the bottleneck to beyond $2e+2$, then we can express

$$\phi_a^{-1}(\alpha_a(x)) = \lim_{n\to\infty} k_a^{-m}(k_a^{-n}(\exp_{e^{1/e}}^n(\exp_{e^{1/e}}^m(\alpha_{e^{1/e}}^{-1}(\psi_a \\ (g_a^{-m}(g_a^{-n}(\exp_a^n(\exp_a^m(x))))))))).$$

We now proceed as in proposition 1. For a given x, we can find an m (possibly negative) such that

$$2e+2 \le \exp_a^m(x) \le a^{2e+2} < 3/2^{2e+2}.$$

By Lemma 1, we have

$$|g_a^{-n}(\exp_a^n(\exp_a^m(x))) - \exp_a^m(x)| < \frac{\delta}{4}.$$

which would indicate that $2e+2-\delta/4 < g_a^{-n}(\exp_a^n(\exp_a^m(x))) < (3/2)^{2e+2}+\delta/4$.

Note that $\psi_a(g_a^{-m}(x)) = \psi_a(x)-m$, and $\alpha_{e^{1/e}}^{-1}(x-m) = \exp_{e^{1/e}}^{-m}(\alpha_{e^{1/e}}^{-1}(x))$. Thus,

$$\exp_{e^{1/e}}^m(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-m}(x)))) = \alpha_{e^{1/e}}^{-1}(\psi_a(x)).$$

Although $\alpha_{e^{1/e}}(x)$ and $\psi_a(x)$ are undefined at the fixed points,

$$\lambda(x) = \frac{1}{\alpha'_{e^{1/e}}(x)} \qquad \text{and} \qquad \lambda_a(x) = \frac{1}{\psi'_a(x)}$$

have unique formal power series near the fixed point. Since the power series of λ_a approaches the power series for λ as $a \to e^{1/e}$, we can find an a_0 such that

$$|\alpha_{e^{1/e}}^{-1}(\phi_a(x)) - x| < \frac{\delta}{4}$$

for all $e < x < 2e+2-\delta/4$, and $e^{1/e} < a < a_0$. Since $g_a^{-n}(\exp_a^n(\exp_a^m(x)))$ will be larger than $2e+2-\delta/4$, we have

$$|\exp_{e^{1/e}}^m(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-m}(g_a^{-n}(\exp_a^n(\exp_a^m(x)))))))-g_a^{-n}(\exp_a^n(\exp_a^m(x)))| < \frac{\delta}{4},$$

so

$$|\exp_{e^{1/e}}^m(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-m}(g_a^{-n}(\exp_a^n(\exp_a^m(x))))))) - \exp_a^m(x)| < \frac{\delta}{2}.$$

Since $\exp_{e^{1/e}}^m(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-m}(g_a^{-n}(\exp_a^n(\exp_a^m(x))))))) > 2e+2-\delta/2$, we can use Lemma 1 to show that

$$|k_a^{-n}(\exp_{e^{1/e}}^{n+m}(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-n-m}(\exp_a^n(\exp_a^m(x)))))) \\ -\exp_{e^{1/e}}^m(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-n-m}(\exp_a^n(\exp_a^m(x))))))| < \frac{\delta}{4},$$

so

$$|k_a^{-n}(\exp_{e^{1/e}}^{n+m}(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-n-m}(\exp_a^n(\exp_a^m(x))))))) - \exp_a^m(x)| < \frac{3\delta}{4}.$$

If we let $y = \exp_a^m(x)$ and $h(y) = k_a^{-n}(\exp_{e^{1/e}}^{n+m}(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-n-m}(\exp_a^n(y))))))$, we find that $h(x)$ does not depend on m, and in fact uniformly converges to x as $a \to e^{1/e}$. Thus, we can use Lemma 2 with $f_a = k_a$ and $g_a = a^x$, with $N = (3/2)^{2e+2} + 1$, to show that for a sufficiently close to $e^{1/e}$,

$$|k_a^{-m}(k_a^{-n}(\exp_{e^{1/e}}^n(\exp_{e^{1/e}}^m(\alpha_{e^{1/e}}^{-1}(\psi_a(g_a^{-m}(g_a^{-n}(\exp_a^n(\exp_a^m(x)))))))))) - x| < \delta$$

Taking the limit as $n \to \infty$ gives us our result. □

COROLLARY 1. *The function* $\alpha_a(x)$ *approaches* $\alpha_{e^{1/e}}$ *for* $x < e$ *as* $a \to e^{1/e}$. *Hence, this tetration passes the* continuity requirement.

Proof. Both $\alpha_a(x)$ and $\alpha_{e^{1/e}}$ are normalized so that $\alpha_a(1) = 0$ and $\alpha_{e^{1/e}}(1) = 0$. It is clear that for a closed interval not including e, $\phi_a(x)$ with the same normalization will uniformly approach $\alpha_{e^{1/e}}$, since $k_a(x)$ uniformly approaches $e^{x/e}$, and the interval would not include the bottleneck region. But proposition 1 shows that $\alpha_a(x)$ is uniformly close to $\phi_a(x)$, so $\alpha_a(x)$ uniformly approaches $\alpha_{e^{1/e}}$ on any closed interval not including e. □

7. Conclusion

By embracing the fact that the solution to Abel's equation is not unique, we have formed a metric allowing us to measure distances between solutions. We also found one solution that is particularly easy to calculate, that is more or less "in between" other proposed solutions. We were able to extend this solution to solve the fractional iterates of a^x, and proved that this new solution approaches the established solution as a approaches the critical value of $e^{1/e}$. Hence, we have a viable new solution to the tetration problem.

We still have yet to study how this new solution behaves in the complex plane. This can be explored in a future paper.

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William Paulsen
Department of Mathematics and Statistics
Arkansas State University
Jonesboro, AR 72401, USA
E-mail: `wpaulsen@astate.edu`